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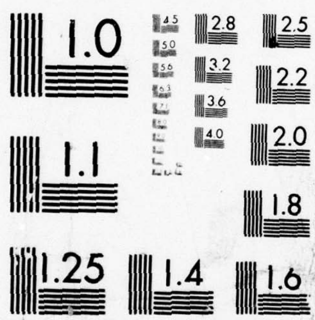

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THE USE OF STRIP THEORY IN THE DYNAMICS OF DEFORMABLE AIRCRAFT

by

D. L. Woodcock

SUMMARY

A detailed formulation of the equations of motion of a deformable aircraft is given. The development is from Lagrange's equations for an inertial frame, and is made in terms of the position, orientation, force and inertia properties of narrow strips of the aircraft which lie fore and aft in the unperturbed state. The latter is one of constant linear velocity and zero angular velocity. Particular account is taken of the deformation and loading in the unperturbed state.

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1 INTRODUCTION

There are two ways of trying to understand a complicated problem. One is, as it were, to stand back and see the thing as a whole without being confused by all the detail - to look at the wood and not the trees. The other is to pick on a basic unit - the tree of the phrase - and look at that first. In each case one has ultimately of course to bring in the details - to look closer at the wood or to build up the wood from the trees. In Ref 1 the first approach was adopted; but a recent note by Baldock[5] showed that there are some, and they belong to the category of those who do it rather than those who show you how to do it, who want to use the other route. The consequence in this paper, which is a considerable generalisation of what Baldock did, written in a notation which harmonises with that of Ref 1 (and indeed of Refs 3 and 4).

2 THE DATUM MOTION

We wish to consider perturbations from a datum motion. What then should be the unperturbed state? How general a motion should it be? What restrictions should be placed upon it? One view would be to select it to suit flight conditions encountered in practice; but alternatively one could choose it to ensure the minimum of complication in the analysis, and then, if necessary, generalise when required. We have taken the latter approach and in particular have said that the datum motion shall be such that the shape of the aircraft remains constant throughout. In particular, this ensures considerable simplification of the aerodynamics and kinematics of the system. To ensure, in general, that there is no change of shape during the datum motion we therefore specify that:

(i) the aircraft's mass and mass distribution are constant,

(ii) the atmosphere and the earth's gravitational field are uniform, and

(iii) the motion is one of constant linear velocity and zero angular velocity relative to earth fixed axes.

We also assume that the aircraft is symmetric in the unperturbed state.

Consequently, we take a set of constant-velocity axes $O_c x_c y_c z_c$ which has zero angular velocity and constant linear velocity V_f in a direction in the plane $O_c x_c z_c$, and is such that it is fixed in the aircraft, $O_c x_c$ being fore and aft, throughout the datum motion. Thus the datum motion is defined by:

$$\begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} = V_f \begin{bmatrix} \cos(\Theta_f - \gamma_f) \\ 0 \\ \sin(\Theta_f - \gamma_f) \end{bmatrix} \tag{2-1}$$

$$\begin{bmatrix} \Phi_f \\ \Theta_f \\ \Psi_f \end{bmatrix} = \begin{bmatrix} 0 \\ \Theta_f \\ 0 \end{bmatrix} \tag{2-2}$$

where Θ_f is the angle of inclination, γ_f is the angle of climb, the angle of bank is zero; and, with no loss of generality for our purpose, we have taken the nose-azimuth angle and the angle of track both also to be zero*. Thus the constant-velocity axes coincide, during the datum motion, with the body-fixed axes used to define the attitude of the aircraft. One could choose the latter axes to be the principal axes of inertia of the aircraft and consequently obtain some simplification in the expressions for the reversed effective and generalised gravitational forces at the expense of having to find out where the principal axes of inertia are (*cf* Appendix B). Another choice, which may appeal to some, would be to take body-fixed axes which, for the particular datum motion being considered, coincided with the body-path axes having the same origin. In this case one would have the angles of climb and inclination equal ($\gamma_f = \Theta_f$).

3 THE SEMI-RIGID MODEL

We assume the aircraft consists of a number of strips each of which is based on a mean line which, in the datum motion, is normal to $O_c y_c$, passes through the strip centre of gravity, and is more or less fore and aft. Each strip is made up of two rigid portions** joined by a hinge at a point on the mean line. The plane separating a strip from an adjacent strip is not necessarily normal to $O_c y_c$, but instead is chosen to be roughly normal to the external surface of the aircraft at their line of intersection. We also arrange that the

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* The three attitude angles (angle of bank, angle of inclination, nose-azimuth angle) define the attitude of the aircraft, *ie* of some axes fixed in the aircraft, relative to normal earth-fixed axes. The two flight-path angles (angle of climb, angle of track) define the direction of flight relative to the same axes (*cf* Ref 3, sections 5 and 6).

** This image includes the case of a single rigid portion as the case where one of the two portions is of zero length.

mean line of a strip does not extrude from the strip through either of its separating planes. Such a strip models a wing (or tailplane or fin) chordwise strip with a control surface; or, with rather more engineer's licence, even an engine pod or a store or a fuselage.

In Ref 1 an intermediate frame of reference, defined by the no-deformation-body-fixed axes $O_n x_n y_n z_n$, was used in the specification of the perturbed position and shape of the aircraft. In the present context it is convenient also to use sets of strip-fixed axes, one set being associated with each strip. Let the strip-fixed axes for a typical strip be $O_{si} x_{si} y_{si} z_{si}$ where $O_{si} x_{si}$ is the mean line mentioned above, the hinge is on $O_{si} x_{si}$, *ie* at, say,

$$\left\{x_{si}^{(s)},\ y_{si}^{(s)},\ z_{si}^{(s)}\right\} = \left\{x_{hsi},\ 0,\ 0\right\} \tag{3-1}$$

and the axes are fixed in one of the rigid portions of the strip such that the hinge axis of rotation is parallel to the direction $O_{si} y_{si}$, and $O_{si} x_{si} z_{si}$ is a mean plane between the planes dividing the strip from the adjacent strips. Thus for a wing section with a trailing edge control one would take O_{si} to be some reference point *such as the quarter chord point*, $O_{si} x_{si}$ to be forward along the chord line, and x_{hsi} to be the (negative) value of x_{si} at the control surface hinge*. We will call the portion of a strip, in which the strip-fixed axes are fixed, the main part; and the other portion, when it exists, the flap (*cf* Appendix B).

As in Ref 1 we represent the perturbation of the aircraft, from the unperturbed position and shape it would have had at the same instant during the datum motion, as being made up of rigid body translations and rotations which move a set of body-fixed axes from coincidence with the constant-velocity axes to coincidence with the no-deformation-body-fixed axes, followed by some further perturbations which we call deformations. The first part of these perturbations is therefore (*cf* Ref 1) made up of the two successive steps:

(i) translations, as a rigid body, $x_1^{(c)}$, $y_1^{(c)}$, $z_1^{(c)}$ in the directions of the respective constant-velocity axes;

(ii) successive rotations ψ, θ, ϕ about the carried axes Oz, Oy, Oz where Oxyz are the above-mentioned body-fixed axes.

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* The control surface hinge and the hinge of the strip model do not necessarily coincide. If the control surface has a swept hinge and the strip is fore and aft they will not.

The second part of the perturbations - the deformations - we write in a somewhat different form from that of Ref 1.

The transformation from the no-deformation-body-fixed axes to the strip-fixed axes for the ith strip is achieved by:

(a) Translations $x_i^{(n)}$, $y_i^{(n)}$, $z_i^{(n)}$ in the directions of the respective no-deformation-body-fixed axes where these translations are compounded of the datum state separation between the strip reference point 0_{si} and the point 0_c, and a deformational contribution, viz:

$$\begin{bmatrix} x_i^{(n)} \\ y_i^{(n)} \\ z_i^{(n)} \end{bmatrix} = \begin{bmatrix} x_{if} \\ y_{if} \\ z_{if} \end{bmatrix} + K \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \tag{3-2}$$

K is a modal matrix whose elements are functions of the strip being considered, and $q_1 \rightarrow q_n$ are the generalised coordinates for the deformational freedoms. For example if, for a wing, the strip reference points are at the quarter chord point, then K describes the deformational shapes of the quarter chord line in the various modes*.

(b) Successive rotations ψ_i, θ_i, ϕ_i, about the carried axes, which are written in a form similar to (3-2):

$$\begin{bmatrix} \phi_i \\ \theta_i \\ \psi_i \end{bmatrix} = \begin{bmatrix} \phi_{if} \\ \theta_{if} \\ \psi_{if} \end{bmatrix} + F \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} . \tag{3-3}$$

Thus F is a modal matrix. It can be thought of as a 'torsional' modal matrix along with K as a 'flexural' modal matrix. The condition that a strip mean line $0_{si}x_{si}$ is normal to 0_cy_c during the datum motion is satisfied by making $\psi_{if} = 0$, and this does not impose any other restriction, as we will show following equation (3-21). For, say, a wing whose reference axis (line of strip reference points) is parallel to the plane $0_nx_ny_n$ (*ie* z_{if} = constant), whose

* All the values of K can be thought of as a three-dimensional array of numbers. It is this array which describes, in this case, the deformational shapes of the quarter chord line.

only flexibility is in torsion about the reference axis (*ie* the matrix K is zero for all wing strips), and whose strips are fore and aft (*ie* $O_{si}x_{si}$ parallel to $O_nx_nz_n$) during the datum motion, ϕ_{if} and ψ_{if} will be zero and θ_{if} will be the sum of the strip jig 'incidence' and the strip 'incidence' due to twist in the datum motion*.

In addition to the displacements and deformations represented by these two axes transformations we have deformation in the strip itself. The main part of the strip, by definition, cannot be displaced relative to the strip-fixed axes, but the flap part can. We therefore have on the flap part of the strip**

$$\begin{bmatrix} x_{si}^{(s)} \\ y_{si}^{(s)} \\ z_{si}^{(s)} \end{bmatrix} \approx \begin{bmatrix} x_{hsi} \\ 0 \\ 0 \end{bmatrix} + P_{\delta_{if}}^{T} \left\{ I + (\delta_i - \delta_{if}) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} x_{sie} - x_{hsi} \\ y_{sie} \\ z_{sie} \end{bmatrix} \qquad \ldots\ldots \text{(3-4)}$$

where δ_i is the angle of flap rotation, δ_{if} is the flap rotation in the datum motion, x_{sie}, etc are the values of x_{si}, etc, when there is no flap rotation, and $P_{\delta_{if}}$ is the axes transformation matrix (attitude deviation matrix) for the single rotation δ_{if} about $O_{si}y_{si}$ which is given by†

$$P_{\delta_{if}} = \begin{bmatrix} \cos \delta_{if} & 0 & -\sin \delta_{if} \\ 0 & 1 & 0 \\ \sin \delta_{if} & 0 & \cos \delta_{if} \end{bmatrix} . \qquad \text{(3-5)}$$

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* Strictly speaking these angles should be called angles of inclination (in the no-deformation-body-fixed axes reference frame) rather than angles of incidence (*cf* Ref 3, sections 5 and 6).

** We have used the general symbol δ, specified by Hopkin (Ref 3, section 8) for a motivator rotation rather than decide the primary purpose of our flap (say to produce a rolling, pitching or yawing moment) and take the appropriate symbol (ξ, η or ζ).

† Thus $\dfrac{d}{d\delta_{if}}(P_{\delta_{if}}) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} P_{\delta_{if}}$.

(For a general rotation the axes transformation matrix S is of the form $S = R_\phi P_\theta Y_\psi$ - *cf* Ref 4, Appendix A.) The above expression (3-4), assumes that $\delta_i - \delta_{if}$ is small. Putting $\delta_i = \delta_{if}$, as in the datum motion, we therefore find that*, on the flap,

$$\begin{bmatrix} x_{sif}^{(us)} \\ y_{sif}^{(us)} \\ z_{sif}^{(us)} \end{bmatrix} = \begin{bmatrix} x_{hsi} \\ 0 \\ 0 \end{bmatrix} + P_{\delta_{if}}^T \begin{bmatrix} x_{sie} - x_{hsi} \\ y_{sie} \\ z_{sie} \end{bmatrix} . \qquad (3\text{-}6)$$

The perturbation in the flap rotation is related to the generalised deformational coordinates by a flap modal vector f :

$$\delta_i - \delta_{if} = f^T \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \qquad (3\text{-}7)$$

and so, on the flap

$$\begin{bmatrix} x_{si}^{(s)} \\ y_{si}^{(s)} \\ z_{si}^{(s)} \end{bmatrix} \approx \begin{bmatrix} x_{sif}^{(us)} \\ y_{sif}^{(us)} \\ z_{sif}^{(us)} \end{bmatrix} + P_{\delta_{if}}^T \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{sie} - x_{hsi} \\ y_{sie} \\ z_{sie} \end{bmatrix} f^T \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} . \qquad (3\text{-}8)$$

Combining the successive deformations (equations (3-2), (3-3) and (3-8)) we find that, for a point on the flap, its position relative to the origin of the no-deformation-body-fixed axes is given by

* The superscript (us) is here introduced to show that these are the coordinates referred to the unperturbed (datum motion) orientation of the strip-fixed axes.

$$\begin{bmatrix} x_{ni}^{(n)} \\ y_{ni}^{(n)} \\ z_{ni}^{(n)} \end{bmatrix} \approx \begin{bmatrix} x_{if} \\ y_{if} \\ z_{if} \end{bmatrix} + K \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$+ S_{\phi_i}^T \left\{ \begin{bmatrix} x_{sif}^{(us)} \\ y_{sif}^{(us)} \\ z_{sif}^{(us)} \end{bmatrix} + P_{\delta_{if}}^T \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{sie} - x_{hsi} \\ y_{sie} \\ z_{sie} \end{bmatrix} f^T \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right\} \quad \ldots\ldots \text{(3-9)}$$

where S_{ϕ_i} $(= R_{\phi_i} P_{\theta_i} Y_{\psi_i})$ is the axes transformation matrix for the change in orientation from the no-deformation-body-fixed axes to the strip-fixed axes (*cf* Ref 4, Appendix A)

$$R_{\phi_i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_i & \sin\phi_1 \\ 0 & -\sin\phi_i & \cos\phi_1 \end{bmatrix} \quad \text{(3-10)}$$

$$P_{\theta_i} = \begin{bmatrix} \cos\theta_i & 0 & -\sin\theta_i \\ 0 & 1 & 0 \\ \sin\theta_i & 0 & \cos\theta_i \end{bmatrix} \quad \text{(3-11)}$$

and

$$Y_{\psi_i} = \begin{bmatrix} \cos\psi_i & \sin\psi_i & 0 \\ -\sin\psi_i & \cos\psi_i & 0 \\ 0 & 0 & 1 \end{bmatrix} . \quad \text{(3-12)}$$

Writing

$$\begin{bmatrix} \alpha_i \\ \beta_i \\ \gamma_i \end{bmatrix} = Q_{\phi_{if}} \begin{bmatrix} \phi_i - \phi_{if} \\ \theta_i - \theta_{if} \\ \psi_i - \psi_{if} \end{bmatrix} = Q_{\phi_{if}} F \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \quad \text{(3-13)}$$

where

$$Q_{\phi_{if}} = \begin{bmatrix} 1 & 0 & -\sin\theta_{if} \\ 0 & \cos\phi_{if} & \sin\phi_{if}\cos\theta_{if} \\ 0 & -\sin\phi_{if} & \cos\phi_{if}\cos\theta_{if} \end{bmatrix} \tag{3-14}$$

it can be shown that*

$$S_{\phi_i} \approx \left(I - A_{\alpha_i}\right) S_{\phi_{if}} \tag{3-15}$$

where

$$A_{\alpha_i} = \begin{bmatrix} 0 & -\gamma_i & \beta_i \\ \gamma_i & 0 & -\alpha_i \\ -\beta_i & \alpha_i & 0 \end{bmatrix} . \tag{3-16}$$

Thus to first order, making use of the fact that, for matrices of the latter type,

$$A_\phi \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -A_x \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \tag{3-17}$$

we have

$$\begin{bmatrix} x_{ni}^{(n)} \\ y_{ni}^{(n)} \\ z_{ni}^{(n)} \end{bmatrix} \approx \begin{bmatrix} x_{if} \\ y_{if} \\ z_{if} \end{bmatrix} + S_{\phi_{if}}^T \begin{bmatrix} x_{sif}^{(us)} \\ y_{sif}^{(us)} \\ z_{sif}^{(us)} \end{bmatrix}$$

$$+ \left\{ K - S_{\phi_{if}}^T A_{x_{sif}^{(us)}} Q_{\phi_{if}} F + S_{\phi_{if}}^T P_{\delta_{if}}^T \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{sie} - x_{hsi} \\ y_{sie} \\ z_{sie} \end{bmatrix} f^T \right\} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

...... (3-18)

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* Thus the angles α_i ... are an approximation to the Euler angles which produce the orientation transformation from the unperturbed position of the strip-fixed axes to their perturbed position when the perturbations ϕ, θ, ψ are zero (*cf* equation (4-40)).

This expression gives the position of a particle on the flap part of a strip relative to the origin of the no-deformation-body-fixed axes and resolved along these axes. For a particle on the main part of a strip one merely takes the modal vector f to be zero. This equation compares with expression

$$\begin{bmatrix} x_n^{(n)} \\ y_n^{(n)} \\ z_n^{(n)} \end{bmatrix} = \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + R \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \tag{3-19}$$

used* in Ref 1 (equation (1)). Thus for the strip model, the modal matrix R has the form:

$$R = K - S^T_{\phi_{if}} A_{x^{(us)}_{sif}} Q_{\phi_{if}} F + S^T_{\phi_{if}} P^T_{\delta_{if}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{sie} - x_{hsi} \\ y_{sie} \\ z_{sie} \end{bmatrix} f^T \tag{3-20}$$

at the ith strip, and the unperturbed coordinates of a particle, relative to and resolved along the body-fixed axes, in the datum state, are:

$$\begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} = \begin{bmatrix} x_{if} \\ y_{if} \\ z_{if} \end{bmatrix} + S^T_{\phi_{if}} \begin{bmatrix} x^{(us)}_{sif} \\ y^{(us)}_{sif} \\ z^{(us)}_{sif} \end{bmatrix} \tag{3-21}$$

for a particle on the ith strip. If this strip is, as required, in a plane normal to $O_c y_c$ then, for a point on $O_{si} x_{si}$ (*ie* $y_{sif} = z_{sif} = 0$), y_f must be constant. Consequently the 12 term of $S_{\phi_{if}}$ must be zero and so (*cf* equations (3-10), (3-11) and (3-12))

$$\cos \theta_{if} \sin \psi_{if} = 0 \quad . \tag{3-22}$$

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* There is however a difference of some significance: (3-19) is a precise statement of the deformation used in Ref 1 while (3-18), for our present model, is only a first order approximation in the generalised coordinates q_i (*cf* Appendix A).

This is satisfied by $\theta_{if} = (\pi/2)$ or $\psi_{if} = 0$. However, any transformation with $\theta_{if} = (\pi/2)$ is equivalent to another transformation with $\psi_{if} = 0$, and so we will always take

$$\psi_{if} = 0 \quad . \tag{3-23}$$

This makes

$$S_{\phi_{if}} = R_{\phi_{if}} P_{\theta_{if}} = \begin{bmatrix} \cos\theta_{if} & 0 & -\sin\theta_{if} \\ \sin\phi_{if}\sin\theta_{if} & \cos\phi_{if} & \sin\phi_{if}\cos\theta_{if} \\ \cos\phi_{if}\sin\theta_{if} & -\sin\phi_{if} & \cos\phi_{if}\cos\theta_{if} \end{bmatrix} \quad . \qquad \ldots\ldots \; (3\text{-}24)$$

There is one other expression for the position of a particle on a typical (the ith) strip that we may require, and that is for its position relative to the origin of the constant-velocity axes and resolved along those axes. This is easily seen to be

$$\begin{bmatrix} x_c^{(c)} \\ y_c^{(c)} \\ z_c^{(c)} \end{bmatrix} \approx \begin{bmatrix} x_{if} \\ y_{if} \\ z_{if} \end{bmatrix} + S^T_{\phi_{if}} \begin{bmatrix} x^{(us)}_{sif} \\ y^{(us)}_{sif} \\ z^{(us)}_{sif} \end{bmatrix} + R \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix} - \left(A_{x_{if}} + S^T_{\phi_{if}} A_{x^{(us)}_{sif}} S_{\phi_{if}} \right) \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \qquad \ldots\ldots \; (3\text{-}25)$$

where R is given above (equation (3-20)).

In this section, we have introduced the generalised coordinates $q_1 \rightarrow q_n$. To complete the set we take $\{x_1^{(c)}\ y_1^{(c)}\ z_1^{(c)}\ \phi\ \theta\ \psi\}$ as the other six generalised coordinates $\{q_{n+1}, \ldots, q_{n+6}\}$ and subsequently we will use either notation capriciously.

4 THE AERODYNAMIC FORCES

4.1 The velocities

The main reason for the adoption of the strip model described in section 3 is of course to facilitate the use of strip theory aerodynamics so beloved by many. The air forces on a strip will depend on the velocity of the air relative to the strip. The velocity of a particle referred to the strip-fixed axes is

$$\begin{bmatrix} u_{mi}^{(s)} \\ v_{mi}^{(s)} \\ w_{mi}^{(s)} \end{bmatrix} = \begin{bmatrix} \dot{x}_{si}^{(s)} \\ \dot{y}_{si}^{(s)} \\ \dot{z}_{si}^{(s)} \end{bmatrix} + S_{\phi_i} \left\{ \left(\dot{S}^T_{\phi_i} + S\dot{S}^T S^T_{\phi_i} \right) \begin{bmatrix} x_{si}^{(s)} \\ y_{si}^{(s)} \\ z_{si}^{(s)} \end{bmatrix} + \begin{bmatrix} \dot{x}_i^{(n)} \\ \dot{y}_i^{(n)} \\ \dot{z}_i^{(n)} \end{bmatrix} \right.$$

$$\left. + S\dot{S}^T \begin{bmatrix} x_i^{(n)} \\ y_i^{(n)} \\ z_i^{(n)} \end{bmatrix} + S \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix} + S \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} \right\} \, .$$

...... (4-1)

This expression can be obtained either by taking the expression for the position of a particle, relative to the origin of the normal earth-fixed axes and resolved along the constant-velocity axes, differentiating, and then premultiplying the result by $S_{\phi_i}S$ to refer the velocity to the strip-fixed axes; or by adding together the various relative velocities making use of the facts that (*cf* Ref 4, Appendix A):

(i) If some axes $O_a x_a y_a z_a$ have angular velocity $\left\{p_b^{(a)} q_b^{(a)} r_b^{(a)}\right\}$ relative to axes $O_b x_b y_b z_b$ and resolved along $O_a x_a y_a z_a$, then the linear velocity of a point relative to O_b resolved along the a-axes is

$$\left(ID + A_{p_b^{(a)}} \right) \begin{bmatrix} x_a \\ y_a \\ z_a \end{bmatrix} + S_{\phi_{ab}} \begin{bmatrix} \dot{x}_{ab}^{(b)} \\ \dot{y}_{ab}^{(b)} \\ \dot{z}_{ab}^{(b)} \end{bmatrix} \qquad (4\text{-}2)$$

where ϕ_{ab} etc are the standard Euler rotations which transform to the a-directions from the b-directions, and $x_{ab}^{(b)}$ etc are the coordinates of 0_a relative to 0_b and resolved along the b-axes.

(ii) In the situation described in (i)

$$\begin{bmatrix} p_b^{(a)} \\ q_b^{(a)} \\ r_b^{(a)} \end{bmatrix} = Q_{\phi_{ab}} \begin{bmatrix} \dot{\phi}_{ab} \\ \dot{\theta}_{ab} \\ \dot{\psi}_{ab} \end{bmatrix} \tag{4-3}$$

(*cf* equation (3-14)), and

$$\begin{aligned} A_{p_b^{(a)}} &= S_{\phi_{ab}} \dot{S}^T_{\phi_{ab}} \\ &= - \dot{S}_{\phi_{ab}} S^T_{\phi_{ab}} \end{aligned} \tag{4-4}$$

and (iii) If $\{x\ y\ z\}$ is any vector and S any axes transformation matrix then*

$$A_{\left(S \begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)} = S A_x S^T \quad . \tag{4-5}$$

In equation (4-1) $\left\{x_{si}^{(s)}\ y_{si}^{(s)}\ z_{si}^{(s)}\right\}$ is given by equation (3-8), f being put equal to zero for a particle not on the flap portion of the strip, $\{\phi_i\ \theta_i\ \psi_i\}$ by equation (3-3), and $\left\{x_i^{(n)}\ y_i^{(n)}\ z_i^{(n)}\right\}$ by equation (3-2). Substituting these expressions, and making use of (3-15), (3-17) and (4-5), we have

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* (4-5) is of course, also true with S replaced by S^T, *ie*

$$A_{\left(S^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)} = S^T A_x S \quad .$$

$$\begin{bmatrix} u_{mi}^{(s)} \\ v_{mi}^{(s)} \\ w_{mi}^{(s)} \end{bmatrix} \approx S_{\phi_{if}} \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} + S_{\phi_{if}} A_{u_f} S^T_{\phi_{if}} Q_{\phi_{if}} F \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

$$+ \left\{ S_{\phi_{if}} K - A_{x_{sif}^{(us)}} Q_{\phi_{if}} F + P^T_{\delta_{if}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{sie} - x_{hsi} \\ y_{sie} \\ z_{sie} \end{bmatrix} f^T \right\} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$+ S_{\phi_{if}} \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix} + S_{\phi_{if}} A_{u_f} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} - \left(A_{x_{sif}^{(us)}} S_{\phi_{if}} + S_{\phi_{if}} A_{x_{if}} \right) \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} .$$

...... (4-6)

The matrix factor $S^T_{\phi_{if}} Q_{\phi_{if}}$, in the second term, has, consequent upon (3-14), (3-23) and (3-24), the particularly simple form

$$S^T_{\phi_{if}} Q_{\phi_{if}} = \begin{bmatrix} \cos\theta_{if} & 0 & 0 \\ 0 & 1 & 0 \\ -\sin\theta_{if} & 0 & 1 \end{bmatrix} . \qquad (4\text{-}7)$$

It will also be noticed that the term in the $\{\ \}$ is $S_{\phi_{if}} R$ (*cf* equation (3-20)). The square of the resultant speed of the strip reference point is easily seen to be

$$V_i^2 \equiv \left(\begin{bmatrix} u_{mi}^{(s)} & v_{mi}^{(s)} & w_{mi}^{(s)} \end{bmatrix} \begin{bmatrix} u_{mi}^{(s)} \\ v_{mi}^{(s)} \\ w_{mi}^{(s)} \end{bmatrix}\right) \begin{bmatrix} x_{sif}^{(us)} \\ y_{sif}^{(us)} \\ z_{sif}^{(us)} \end{bmatrix} = 0$$

$$= \begin{bmatrix} u_f & v_f & w_f \end{bmatrix} \left(\begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} + 2 \left\{ \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix} - A_{x_{if}} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} + K \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} \right\} \right)$$

$$= V_f^2 \left(1 + \frac{2}{V_f^2} \begin{bmatrix} u_f & v_f & w_f \end{bmatrix} \left\{ \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix} - A_{x_{if}} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} + K \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} \right\} \right) \tag{4-8}$$

4.2 The forces on a strip

4.2.1 Their representation

Consider a vortex whose axis lies along Oy in a frame of reference Oxyz which is moving with linear velocity $\{u\ v\ w\}$, referred to the same axes, and zero angular velocity, through a fluid. Then the force on that vortex is

$$-\rho A_u \begin{bmatrix} 0 \\ \Gamma \\ 0 \end{bmatrix} \equiv \rho \begin{bmatrix} 0 & 0 & \Gamma \\ 0 & 0 & 0 \\ -\Gamma & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{4-9}$$

referred to Oxyz, where Γ is the vortex strength. In applying strip theory we represent the strip by a set of bound vortices* whose axes are parallel to

* Plus also a semi-infinite layer of free vortices which move with the fluid and so have no force exerted upon them.

$O_{si}y_{si}$. The strengths of these vortices will be functions of the boundary condition and so of the velocity of the strip normal to its surface. We will approximate to this on the main part of the strip by $w_{mi}^{(s)}$ at points on $O_{si}x_{si}$; and on the flap part of the strip by the velocity normal to $O_{si}y_{si}$ and to the line (*cf* equations (3-6) and (3-8)) given by $y_{sie} = z_{sie} = 0$, *ie* the line

$$\begin{bmatrix} x_{si}^{(s)} \\ y_{si}^{(s)} \\ z_{si}^{(s)} \end{bmatrix} \approx \begin{bmatrix} x_{hsi} \\ 0 \\ 0 \end{bmatrix} + (x_{sie} - x_{hsi}) \left\{ \begin{bmatrix} \cos\delta_{if} \\ 0 \\ -\sin\delta_{if} \end{bmatrix} - \begin{bmatrix} \sin\delta_{if} \\ 0 \\ \cos\delta_{if} \end{bmatrix} f^T \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right\} . \qquad \ldots\ldots (4\text{-}10)$$

This latter velocity is therefore the last element of $P_{\delta_i} \begin{bmatrix} u_{mi}^{(s)} \\ v_{mi}^{(s)} \\ w_{mi}^{(s)} \end{bmatrix}$ at points on (4-10), which to the desired accuracy is (*cf* equations (3-5) and (3-7))

$$[0 \quad 0 \quad 1] \left\{ P_{\delta_{if}} \begin{bmatrix} u_{mi}^{(s)} \\ v_{mi}^{(s)} \\ w_{mi}^{(s)} \end{bmatrix} + \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} P_{\delta_{if}} \begin{bmatrix} u_{mi}^{(s)} \\ v_{mi}^{(s)} \\ w_{mi}^{(s)} \end{bmatrix} f^T \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right\}$$

$$\approx [0 \quad 0 \quad 1] \, P_{\delta_{if}} \begin{bmatrix} u_i^{(s)} \\ v_i^{(s)} \\ w_i^{(s)} \end{bmatrix} - \left\{ x_{hsi} \cos\delta_{if} + (x_{sie} - x_{hsi}) \right\} [0 \quad 1 \quad 0] \begin{bmatrix} p_i^{(s)} \\ q_i^{(s)} \\ r_i^{(s)} \end{bmatrix}$$

$$+ [1 \quad 0 \quad 0] \, P_{\delta_{if}} S_{\phi_{if}} \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} f^T \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} - (x_{sie} - x_{hsi}) f^T \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} \qquad \ldots\ldots (4\text{-}11)$$

where $\left\{u_i^{(s)}\ v_i^{(s)}\ w_i^{(s)}\right\}$ and $\left\{p_i^{(s)}\ q_i^{(s)}\ r_i^{(s)}\right\}$ are given below. It will be seen that $w_{mi}^{(s)}$ at a point on $O_{si}x_{si}$ on the main part of the strip is the same expression with f and δ_{if} both zero. The abbreviations used are:

$$\begin{bmatrix} u_i^{(s)} \\ v_i^{(s)} \\ w_i^{(s)} \end{bmatrix} \equiv S_{\phi_{if}} \left\{ \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} + A_{u_f} S_{\phi_{if}}^T Q_{\phi_{if}} F \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + K \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} + \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix} + A_{u_f} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} - A_{x_{if}} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \right\} \qquad \text{......} \quad (4\text{-}12)$$

$$\begin{bmatrix} p_i^{(s)} \\ q_i^{(s)} \\ r_i^{(s)} \end{bmatrix} \equiv Q_{\phi_{if}} F \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} + S_{\phi_{if}} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} . \qquad (4\text{-}13)$$

The boundary condition is therefore made up of certain multiples of four basic conditions (at the points described above):

normal velocity of air $= 1$

" $= x_{sie}$

" $= H(x_{hsi} - x_{sie})$

" $= (x_{hsi} - x_{sie})H(x_{hsi} - x_{sie})$

where H is the Heaviside step function.

The multiples are:

$w_i^{(s)}$

$q_i^{(s)}$

$$u_i^{(s)} \sin \delta_{if} + w_i^{(s)}(\cos \delta_{if} - 1) + q_i^{(s)} x_{hsi}(1 - \cos \delta_{if}) + \left(u_{if}^{(us)} \cos \delta_{if} - w_{if}^{(us)} \sin \delta_{if}\right)(\delta_i - \delta_{if})$$

and

$\dot{\delta}_i$

(us denotes unperturbed-strip-fixed axes).

The strength of the vortex will therefore be a function, for a given datum state, of these four scalars* and of its position relative to the wake and to the points where the boundary condition is applied. Thus we can say equivalently that the vortex strength is a function of

$$u_i^{(s)} \ , \ w_i^{(s)} \ , \ q_i^{(s)} \ \text{and} \ \delta_i \ .$$

Thus, for small perturbations the strength of a vortex has the form

$$\Gamma \approx \Gamma_f + \Gamma_u\left(u_i^{(s)} - u_{if}^{(us)}\right) + \Gamma_w\left(w_i^{(s)} - w_{if}^{(s)}\right) + \Gamma_q q_i^{(s)} + \Gamma_\delta(\delta_i - \delta_{if}) \quad \ldots\ldots \ (4\text{-}14)$$

where Γ_f, Γ_u, etc are functions** of $u_{if}^{(us)}$, $w_{if}^{(us)}$, and δ_{if}. The velocity of the vortex relative to the fluid will be in the direction of the boundary line. The enforcement of the boundary condition ensures that there is no relative velocity normal to this line. The force on the vortex is therefore normal to this line. The relative velocity of the vortex, and consequently the force on the vortex will be a function of the same four scalars as the vortex strength (*cf* equation (4-14)). Thus the local aerodynamic force at a point on the strip will have the form

$$\begin{bmatrix} e_i^{(s)} \\ f_i^{(s)} \\ g_i^{(s)} \end{bmatrix} = \chi_{if}\begin{bmatrix} \sin\delta_{if} \\ 0 \\ \cos\delta_{if} \end{bmatrix} + \begin{bmatrix} \chi_{iu}\sin\delta_{if} & \chi_{iw}\sin\delta_{if} & \chi_{iq}\sin\delta_{if} & \chi_{i\delta}\sin\delta_{if} + \chi_{if}\cos\delta_{if} \\ 0 & 0 & 0 & 0 \\ \chi_{iu}\cos\delta_{if} & \chi_{iw}\cos\delta_{if} & \chi_{iq}\cos\delta_{if} & \chi_{i\delta}\cos\delta_{if} - \chi_{if}\cos\delta_{if} \end{bmatrix}$$

$$\times \begin{bmatrix} u_i^{(s)} - u_{if}^{(us)} \\ w_i^{(s)} - w_{if}^{(us)} \\ q_i^{(s)} \\ \delta_i - \delta_{if} \end{bmatrix} . \quad (4\text{-}15)$$

---

* We are taking the view that the strength at an instant is determined by the boundary condition (and a finite number of its derivatives with respect to time) at the same instant.

** Γ_u, Γ_w, Γ_{q_1}, Γ_δ may contain terms involving the differential operator $D = \frac{d}{dt}$.

where δ_{if} (in the trig. functions) and χ_{if} (in the perturbation part) are put equal to zero off the flap portion, and the coefficients χ_{iu}, χ_{iw}, χ_{iq}, $\chi_{i\delta}$ may involve the differential operator D. In addition, there will be a leading edge 'suction' force, as will be seen if one imagines the infinitely thin leading edge, of our aerofoil model, as the limit of a rounded edge. The Kutta-Joukowski theorem tells us that for a two-dimensional aerofoil moving with constant and uniform velocity the overall force on the aerofoil is normal to the direction of motion. The leading edge 'suction' force, which has only a component in the $O_{si}x_{si}$ direction, can be written as*

$$X_{i0}^{(s)} = X_{if0}^{(2)} + \begin{bmatrix} \hat{X}_{i\dot{x}0}^{(2)} & \hat{X}_{i\dot{z}0}^{(2)} & \hat{X}_{i\dot{\theta}0}^{(2)} & X_{i\delta 0}^{(2)} \end{bmatrix} \begin{bmatrix} u_i^{(s)} - u_{if}^{(s)} \\ w_i^{(s)} - w_{if}^{(s)} \\ q_i^{(s)} \\ \delta_i - \delta_{if} \end{bmatrix} \qquad (4\text{-}16)$$

and so it can easily be shown that

$$X_{if0}^{(2)} = -\frac{w_{if}^{(us)}}{u_{if}^{(us)}}\left(\sum{}' \chi_{if} + \sum{}'' \chi_{if}\cos\delta_{if}\right) - \sum{}'' \chi_{if}\sin\delta_{if} \qquad (4\text{-}17)$$

$$\hat{X}_{i\dot{x}0}^{(2)} = \frac{w_{if}^{(us)}}{\left(u_{if}^{(us)}\right)^2}\left\{\sum{}' \chi_{if} + \sum{}'' \chi_{if}\cos\delta_{if}\right\} - \frac{w_{if}^{(us)}}{u_{if}^{(us)}}\left\{\sum{}' \chi_{iu} + \sum{}'' \chi_{iu}\cos\delta_{if}\right\}$$
$$- \sum{}'' \chi_{iu}\sin\delta_{if} + O(D) \qquad (4\text{-}18)$$

$$\hat{X}_{i\dot{z}0}^{(2)} = -\frac{1}{u_{if}^{(us)}}\left\{\sum{}' \chi_{if} + \sum{}'' \chi_{if}\cos\delta_{if}\right\} - \frac{w_{if}^{(us)}}{u_{if}^{(us)}}\left\{\sum{}' \chi_{iw} + \sum{}'' \chi_{iw}\cos\delta_{if}\right\}$$
$$- \sum{}'' \chi_{iw}\sin\delta_{if} + O(D) \qquad (4\text{-}19)$$

---

* The (2) superscript indicates two-dimensional values, and the circumflex suprascript has been used to indicate the affinity of these coefficients with the body-fixed axes coefficients of Ref 1.

$$\hat{X}^{(2)}_{i\delta 0} = -\sum{}'' \chi_{if} \cos \delta_{if} + \frac{w^{(us)}_{if}}{u^{(us)}_{if}} \sum{}'' \chi_{if} \sin \delta_{if}$$

$$- \frac{w^{(us)}_{if}}{u^{(us)}_{if}} \left\{ \sum{}' \chi_{i\delta} + \sum{}'' \chi_{i\delta} \cos \delta_{if} \right\} - \sum{}'' \chi_{i\delta} \sin \delta_{if} + 0(D)$$

...... (4-20)

where $\sum{}'$ indicates a summation over the main part of the strip, and $\sum{}''$ indicates a summation over the flap part.

The overall forces on a strip are (*cf* (4-16)) written as*, referred to the strip-fixed axes (see Fig 1)

$$\begin{bmatrix} X^{(s)}_i \\ Y^{(s)}_i \\ Z^{(s)}_i \end{bmatrix} \approx \begin{bmatrix} X^{(2)}_{if} \\ 0 \\ Z^{(2)}_{if} \end{bmatrix} + \begin{bmatrix} \hat{X}^{(2)}_{ix} & \hat{X}^{(2)}_{iz} & \hat{X}^{(2)}_{i\theta} & \hat{X}^{(2)}_{i\delta} \\ 0 & 0 & 0 & 0 \\ \hat{Z}^{(2)}_{ix} & \hat{Z}^{(2)}_{iz} & \hat{Z}^{(2)}_{i\theta} & \hat{Z}^{(2)}_{i\delta} \end{bmatrix} \begin{bmatrix} u^{(s)}_i - u^{(us)}_{if} \\ w^{(s)}_i - w^{(us)}_{if} \\ q^{(s)}_i \\ \delta_i - \delta_{if} \end{bmatrix} \qquad (4\text{-}21)$$

and so we find that

$$Z^{(2)}_{if} = \sum{}' \chi_{if} + \sum{}'' \chi_{if} \cos \delta_{if} \qquad (4\text{-}22)$$

$$\hat{Z}^{(2)}_{ix} = \left\{ \sum{}' \chi_{iu} + \sum{}'' \chi_{iu} \cos \delta_{if} \right\} + 0(D) \qquad (4\text{-}23)$$

$$\hat{Z}^{(2)}_{iz} = \left\{ \sum{}' \chi_{iw} + \sum{}'' \chi_{iw} \cos \delta_{if} \right\} + 0(D) \qquad (4\text{-}24)$$

$$\hat{Z}^{(2)}_{i\delta} = \left\{ \sum{}' \chi_{i\delta} + \sum{}' \chi_{i\delta} \cos \delta_{if} - \sum{}'' \chi_{if} \sin \delta_{if} \right\} + 0(D) \qquad (4\text{-}25)$$

---

* The subscript i has been used to indicate a typical strip whereas, in Ref 1, X_i , e.g. was the force due to unit displacement in the ith mode.

$$X_{if}^{(2)} = -\frac{w_{if}^{(us)}}{u_{if}^{(us)}} Z_{if}^{(2)} \tag{4-26}$$

$$\hat{X}_{i\dot{x}}^{(2)} = -\frac{w_{if}^{(us)}}{u_{if}^{(us)}} \hat{Z}_{i\dot{x}}^{(2)} + \frac{w_{if}^{(us)}}{\left(u_{if}^{(us)}\right)^2} Z_{if}^{(2)} + O(D) \tag{4-27}$$

$$\hat{X}_{i\dot{z}}^{(2)} = -\frac{w_{if}^{(us)}}{u_{if}^{(us)}} \hat{Z}_{i\dot{z}}^{(2)} - \frac{1}{u_{if}^{(us)}} Z_{if}^{(2)} + O(D) \tag{4-28}$$

$$\hat{X}_{i\dot{\delta}}^{(2)} = -\frac{w_{if}^{(us)}}{u_{if}^{(us)}} \hat{Z}_{i\dot{\delta}}^{(2)} + O(D) \quad . \tag{4-29}$$

Some further consideration is given to this aspect in section 4.22 where a further relationship (4-50) is obtained (see also equations (4-54) to (4-58). Similarly the overall moments on the strip are written as, referred to the strip-fixed axes*,

$$\begin{bmatrix} L_i^{(s)} \\ M_i^{(s)} \\ N_i^{(s)} \end{bmatrix} \approx \begin{bmatrix} 0 \\ M_{if}^{(2)} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ \hat{M}_{i\dot{x}}^{(2)} & \hat{M}_{i\dot{z}}^{(2)} & \hat{M}_{i\dot{\theta}}^{(2)} & \hat{M}_{i\dot{\delta}}^{(2)} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_i^{(s)} - u_{if}^{(us)} \\ w_i^{(s)} - w_{if}^{(us)} \\ q_i^{(s)} \\ \delta_i - \delta_{if} \end{bmatrix} . \tag{4-30}$$

There will be no contribution to these from the leading edge 'suction', and so we find that

$$M_{if}^{(2)} = -\sum \chi_{if} x_{sie} - \left(\sum\nolimits'' \chi_{if}\right) x_{hsi} (\cos \delta_{if} - 1) \tag{4-31}$$

$$\hat{M}_{i\dot{x}}^{(2)} = -\sum \chi_{iu} x_{sie} - \left(\sum\nolimits'' \chi_{iu}\right) x_{hsi} (\cos \delta_{if} - 1) \tag{4-32}$$

---

* In accordance with our practice the symbols for the moments should also have a subscript s to show that the moments are about the origin of the strip-fixed axes but this has been omitted to avoid confusion with the subsequent use of subscript s to denote structural.

$$\hat{M}^{(2)}_{i\dot{z}} = -\sum \chi_{iw} x_{sie} - \left(\sum{}'' \chi_{iw}\right) x_{hsi}(\cos \delta_{if} - 1) \tag{4-33}$$

$$\hat{M}^{(2)}_{i\dot{\theta}} = -\sum \chi_{iq} x_{sie} - \left(\sum{}'' \chi_{iq}\right) x_{hsi}(\cos \delta_{if} - 1) \tag{4-34}$$

$$\begin{aligned}\hat{M}^{(2)}_{i\delta} = -\sum \chi_{i\delta} x_{sie} &- \left(\sum{}'' \chi_{i\delta}\right) x_{hsi}(\cos \delta_{if} - 1) \\ &+ \left(\sum{}'' \chi_{if}\right) x_{hsi} \sin \delta_{if} \quad .\end{aligned} \tag{4-35}$$

The strip hinge moment can be obtained as a particularisation of this obtained by imagining the reference point (of the strip) to be at the hinge and performing the summations for the flap part only. Thus it is

$$B_i = B^{(2)}_{if} + \begin{bmatrix} B^{(2)}_{i\dot{x}} & B^{(2)}_{i\dot{z}} & B^{(2)}_{i\dot{\theta}} & B^{(2)}_{i\delta} \end{bmatrix} \begin{bmatrix} u_i^{(s)} - u_{if}^{(us)} \\ w_i^{(s)} - w_{if}^{(us)} \\ q_i^{(s)} \\ \delta_i - \delta_{if} \end{bmatrix} \tag{4-36}$$

where

$$\left.\begin{aligned} B^{(2)}_{if} &= -\sum{}'' \chi_{if}(x_{sie} - x_{hsi}) \\ B^{(2)}_{i\dot{x}} &= -\sum{}'' \chi_{iu}(x_{sie} - x_{hsi}) \\ &\vdots \qquad\qquad \vdots \\ B^{(2)}_{i\delta} &= -\sum{}'' \chi_{i\delta}(x_{sie} - x_{hsi}) \end{aligned}\right\} \quad . \tag{4-37}$$

4.2.2 An equivalent form

It is of interest to express the aerodynamic force on a strip in terms of coordinates which describe the position and orientation of the strip relative to its position in the datum motion. The position of the strip reference point,

relative to its position in the datum motion, resolved along the constant-velocity axes is (*cf* section 3 and equation (3-2))

$$\begin{bmatrix} x_{ui}^{(c)} \\ y_{ui}^{(c)} \\ z_{ui}^{(c)} \end{bmatrix} = \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix} + S_\phi^T \begin{bmatrix} x_i^{(n)} \\ y_i^{(n)} \\ z_i^{(n)} \end{bmatrix} - \begin{bmatrix} x_{if} \\ y_{if} \\ z_{if} \end{bmatrix}$$

$$\approx \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix} - A_{x_{if}} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + K \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} . \tag{4-38}$$

The axes transformation matrix, for the change in orientation going from the unperturbed position of the strip-fixed axes to the strip-fixed axes, is

$$S_{\phi_{ui}} = S_{\phi_i} S_\phi S^T_{\phi_{if}} \approx S_{\phi_i} S^T_{\phi_{if}} - S_{\phi_{if}} A_\phi S^T_{\phi_{if}} \tag{4-39}$$

and, so, making use of equations (3-15) and (4-5), the standard Euler rotations which achieve this change in orientation are

$$\begin{bmatrix} \phi_{ui} \\ \theta_{ui} \\ \psi_{ui} \end{bmatrix} \approx Q_{\phi_{if}}{}^F \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + S_{\phi_{if}} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} . \tag{4-40}$$

It follows therefore from equations (4-12) and (4-13) that

$$
\begin{bmatrix} u_i^{(s)} - u_{if}^{(us)} \\ v_i^{(s)} - v_{if}^{(us)} \\ w_i^{(s)} - w_{if}^{(us)} \end{bmatrix} \approx S_{\phi_{if}} \left\{ \begin{bmatrix} \dot{x}_{ui}^{(c)} \\ \dot{y}_{ui}^{(c)} \\ \dot{z}_{ui}^{(c)} \end{bmatrix} + A_{u_f} S_{\phi_{if}}^T \begin{bmatrix} \phi_{ui} \\ \theta_{ui} \\ \psi_{ui} \end{bmatrix} \right\}
$$

$$
= S_{\phi_{if}} \begin{bmatrix} \dot{x}_{ui}^{(c)} \\ \dot{y}_{ui}^{(c)} \\ \dot{z}_{ui}^{(c)} \end{bmatrix} - A_{\phi_{ui}} S_{\phi_{if}} \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} \qquad (4\text{-}41)
$$

and

$$
\begin{bmatrix} p_i^{(s)} \\ q_i^{(s)} \\ r_i^{(s)} \end{bmatrix} \approx \begin{bmatrix} \dot{\phi}_{ui} \\ \dot{\theta}_{ui} \\ \dot{\psi}_{ui} \end{bmatrix} . \qquad (4\text{-}42)
$$

Now, since ψ_{if} is zero, $S_{\phi_{if}}$ is given by equation (3-24), and so

$$
\begin{bmatrix} u_i^{(s)} - u_{if}^{(us)} \\ w_i^{(s)} - w_{if}^{(us)} \\ q_i^{(s)} \end{bmatrix} \approx \begin{bmatrix} \cos(\theta_{if} + \Theta_f - \gamma_f) & -\sin(\theta_{if} + \Theta_f - \gamma_f) & -\cos\phi_{if}\sin(\theta_{if} + \Theta_f - \gamma_f) \\ \cos\phi_{if}\sin(\theta_{if} + \Theta_f - \gamma_f) & \cos\phi_{if}\cos(\theta_{if} + \Theta_f - \gamma_f) & \cos(\theta_{if} + \Theta_f - \gamma_f) \\ 0 & 0 & \frac{1}{V_f} D \end{bmatrix} \begin{bmatrix} \dot{x}_{ui}^{(dp)} \\ \dot{z}_{ui}^{(dp)} \\ V_f\theta_{ui} \end{bmatrix}
$$

$$
+ \sin\phi_{if} \begin{bmatrix} 0 & 0 & \sin(\theta_{if} + \Theta_f - \gamma_f) \\ -1 & -\sin(\theta_{if} + \Theta_f - \gamma_f) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_{ui}^{(dp)} \\ V_f\phi_{ui} \\ V_f\psi_{ui} \end{bmatrix}
$$

...... (4-43)

where we have written

$$\begin{bmatrix} x_{ui}^{(dp)} \\ y_{ui}^{(dp)} \\ z_{ui}^{(dp)} \end{bmatrix} = P^{T}_{\Theta_f-\gamma_f} \begin{bmatrix} x_{ui}^{(c)} \\ y_{ui}^{(c)} \\ z_{ui}^{(c)} \end{bmatrix} . \tag{4-44}$$

The single rotation $(\gamma_f - \Theta_f)$, whose effect is represented by this last equation, has been made so that $x_{ui}^{(dp)}$ is the perturbation of the strip reference point in the direction of the datum motion - the superscript (dp) can be thought of as meaning datum path*. Now since, using (3-24)

$$\begin{bmatrix} u_{if}^{(us)} \\ v_{if}^{(us)} \\ w_{if}^{(us)} \end{bmatrix} = V_f S_{\phi_{if}} \begin{bmatrix} \cos(\Theta_f - \gamma_f) \\ 0 \\ \sin(\Theta_f - \gamma_f) \end{bmatrix}$$

$$= V_f \begin{bmatrix} \cos(\theta_{if} + \Theta_f - \gamma_f) \\ \sin\phi_{if}\sin(\theta_{if} + \Theta_f - \gamma_f) \\ \cos\phi_{if}\sin(\theta_{if} + \Theta_f - \gamma_f) \end{bmatrix} \tag{4-45}$$

we can rewrite (4-43) as

$$\begin{bmatrix} u_i^{(s)} - u_{if}^{(us)} \\ w_i^{(s)} - w_{if}^{(us)} \\ q_i^{(s)} \end{bmatrix} \approx \frac{1}{V_f} \begin{bmatrix} u_{if}^{(us)} & -w_{if}^{(us)}\sec\phi_{if} & -w_{if}^{(us)} \\ w_{if}^{(us)} & u_{if}^{(us)}\cos\phi_{if} & u_{if}^{(us)} \\ 0 & 0 & D \end{bmatrix} \begin{bmatrix} \dot{x}_{ui}^{(dp)} \\ \dot{z}_{ui}^{(dp)} \\ V_f\theta_{ui} \end{bmatrix}$$

$$+ \frac{\tan\phi_{if}}{V_f} \begin{bmatrix} 0 & 0 & w_{if}^{(us)} \\ -V_f\cos\phi_{if} & -w_{if}^{(us)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_{ui}^{(dp)} \\ V_f\phi_{ui} \\ V_f\psi_{ui} \end{bmatrix} .$$

...... (4-46)

---

* See the definition of datum-path earth axes in the Glossary of terms.

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Incidentally, the angles ϕ_{if}, $\theta_{if} + \Theta_f - \gamma_f$ are one of the pairs of angles (*cf* Ref 3, section 6.2) which can be used to specify the incidence of the strip in the datum motion. That is

$$\theta_{if} + \Theta_f - \gamma_f \quad \text{is the 'incidence magnitude'}$$

$$\phi_{if} \quad \text{is the 'incidence-plane angle'} \quad .$$

Thus (4-43) is an expression purely in terms of incidence angles, while (4-46) is its transcription in terms of velocity components and one incidence angle. The latter could have been written purely in terms of velocities at the expense of the inconvenience at a $\sqrt{}$.

From equations (4-21) and (4-46), the overall forces on a strip can then be written as

$$\begin{aligned}
\begin{bmatrix} X_i^{(s)} \\ Y_i^{(s)} \\ Z_i^{(s)} \end{bmatrix} = \begin{bmatrix} X_{if}^{(2)} \\ 0 \\ Z_{if}^{(2)} \end{bmatrix} &+ \begin{bmatrix} \hat{X}_{ix}^{(2)} u_{if}^{(us)} + \hat{X}_{iz}^{(2)} w_{if}^{(us)} & -\hat{X}_{ix}^{(2)} w_{if}^{(us)} \sec\phi_{if} + \hat{X}_{iz}^{(2)} u_{if}^{(us)} \cos\phi_{if} & -\hat{X}_{ix}^{(2)} w_{if}^{(us)} + \hat{X}_{iz}^{(2)} u_{if}^{(us)} + \hat{X}_{i\theta}^{(2)} D & \hat{X}_{i\delta}^{(2)} \\ 0 & 0 & 0 & 0 \\ \hat{Z}_{ix}^{(2)} u_{if}^{(us)} + \hat{Z}_{iz}^{(2)} w_{if}^{(us)} & -\hat{Z}_{ix}^{(2)} w_{if}^{(us)} \sec\phi_{if} + \hat{Z}_{iz}^{(2)} u_{if}^{(us)} \cos\phi_{if} & -\hat{Z}_{ix}^{(2)} w_{if}^{(us)} + \hat{Z}_{iz}^{(2)} u_{if}^{(us)} + \hat{Z}_{i\theta}^{(2)} D & \hat{Z}_{i\delta}^{(2)} \end{bmatrix} \begin{bmatrix} \dot{x}_{ui}^{(dp)}/V_f \\ \dot{z}_{ui}^{(dp)}/V_f \\ \theta_{ui} \\ \delta_i - \delta_{if} \end{bmatrix} \\
&+ \tan\phi_{if} \begin{bmatrix} -\hat{X}_{iz}^{(2)} V_f \cos\phi_{if} & -\hat{X}_{iz}^{(2)} w_{if}^{(us)} & \hat{X}_{ix}^{(2)} w_{if}^{(us)} \\ 0 & 0 & 0 \\ -\hat{Z}_{iz}^{(2)} V_f \cos\phi_{if} & -\hat{Z}_{iz}^{(2)} w_{if}^{(us)} & \hat{Z}_{ix}^{(2)} w_{if}^{(us)} \end{bmatrix} \begin{bmatrix} \dot{y}_{ui}^{(dp)}/V_f \\ \phi_{ui} \\ \psi_{ui} \end{bmatrix} .
\end{aligned}$$

...... (4-47)

(Similar expressions can be obtained for the moments and hinge moment, from equations (4-30) and (4-36), but we will not detail them here.)

To compare with the form taken by some other authors (*eg* Baldock[5]) it is necessary to take the quasi-steady form - all the terms containing the operator D in the above rectangular matrices are omitted and we make use of the relationships (4-27) to (4-29) - in the case where the incidence plane angle ϕ_{if} of the strip in the datum motion is small. The force on the strip then has the magnitude (to a first order approximation).

$$Z_{if}^{(2)}\sqrt{1+\left(\frac{w_{if}^{(us)}}{u_{if}^{(us)}}\right)^2}\ \times$$

$$\times\left\{1+\frac{1}{Z_{if}^{(2)}}\begin{bmatrix}\hat{Z}_{i\dot{x}}^{(2)}u_{if}^{(us)}+\hat{Z}_{i\dot{z}}^{(2)}w_{if}^{(us)} & Z_{if}^{(2)}\dfrac{w_{if}^{(us)}}{u_{if}^{(us)}}+\left(-\hat{Z}_{i\dot{x}}^{(2)}w_{if}^{(us)}+\hat{Z}_{i\dot{z}}^{(2)}u_{if}^{(us)}\right) & \hat{Z}_{i\delta}^{(2)}\end{bmatrix}\begin{bmatrix}\dfrac{\dot{x}_{ui}^{(dp)}}{V_f}\\ \dfrac{\dot{z}_{ui}^{(dp)}}{V_f}+\theta_{ui}\\ \delta_i-\delta_{if}\end{bmatrix}\right\}$$

...... (4-48)

and to the same degree of accuracy, it acts in a direction normal to the direction of motion. Note that in (4-48) $\hat{Z}_{i\dot{x}}^{(2)}$, and similarly with the other coefficients, must be understood as the part of that coefficient which is independent of D . Baldock[5] wrote the same forces as

$$\left(V_f+\dot{x}_{ui}^{(dp)}\right)^2 \times \text{a linear function of } z_{ui}^{(dp)},\ \theta_{ui} \text{ and } (\delta_i-\delta_{if}) \quad .$$

Assuming that he does not dispute the usual conclusions that, in the quasi-steady state, in his notation

$$\left.\begin{aligned}\ell_z &= 0\\ \ell_\alpha &= \ell_{\dot{z}}\end{aligned}\right\} \qquad (4\text{-}49)$$

then his form is equivalent to (4-48) with the proviso that (quasi-steady)

$$\hat{Z}_{i\dot{x}}^{(2)}u_{if}^{(us)}+\hat{Z}_{i\dot{z}}^{(2)}w_{if}^{(us)} = 2Z_{if}^{(2)} \quad . \qquad (4\text{-}50)$$

This of course what one gets from two-dimensional potential flow theory. For example for a two-dimensional aerofoil, in an incompressible fluid, moving with speed $(V_f + \delta V)$ at any incidence α, the lift is proportional to $(V_f + \delta V)^2 \sin(\alpha - \alpha_0)$ where α_0 is the no-lift incidence. Consequently, the normal force on the aerofoil is*

$$
\begin{aligned}
Z &\approx Z_f + Z_{\dot{x}}(u - u_f) + Z_{\dot{z}}(w - w_f) \\
&= C\left\{- V_f^2 \sin\alpha_0 - 2V_f \sin\alpha_0 \,(u - u_f)\right. \\
&\qquad\qquad \left. + V_f \cos\alpha_0 \,(w - w_f)\right\}
\end{aligned}
\tag{4-51}
$$

where C is a constant, and so, since

$$
\left.\begin{aligned}
u_f &= V_f \\
w_f &= 0
\end{aligned}\right\}
\tag{4-52}
$$

$$
\begin{aligned}
Z_{\dot{x}} u_f + Z_{\dot{z}} w_f &= -2CV_f^2 \sin\alpha_0 \\
&= 2Z_f \quad .
\end{aligned}
\tag{4-53}
$$

The relationship (4-50) did not appear from the reflections of section 4.2.1 since there we took account only of our knowledge of the direction of the resultant force on the aerofoil rather than its magnitude. If (4-50) is combined with equations (4-26) to (4-29) we have the relationships

$$
X_{if}^{(2)} = -\frac{w_{if}^{(us)}}{u_{if}^{(us)}} Z_{if}^{(2)} = -\frac{1}{2}\left\{\hat{Z}_{i\dot{x}}^{(2)} w_{if}^{(us)} + \hat{Z}_{i\dot{z}}^{(2)} \frac{\left(w_{if}^{(us)}\right)^2}{u_{if}^{(us)}}\right\} + O(D) \tag{4-54}
$$

$$
\hat{X}_{i\dot{x}}^{(2)} = \frac{1}{2}\left\{-\frac{w_{if}^{(us)}}{u_{if}^{(us)}} \hat{Z}_{i\dot{x}}^{(2)} + \left(\frac{w_{if}^{(us)}}{u_{if}^{(us)}}\right)^2 \hat{Z}_{i\dot{z}}^{(2)}\right\} + O(D) \tag{4-55}
$$

---

* We are taking $\alpha_f = 0$ for simplicity, but this involves no loss in generality. The same relationship (4-53) is obtained for any α_f .

$$\hat{X}^{(2)}_{i\dot{z}} = \frac{1}{2}\left\{- 3\,\frac{w^{(us)}_{if}}{u^{(us)}_{if}}\,\hat{Z}^{(2)}_{i\dot{z}} - \hat{Z}^{(2)}_{i\dot{x}}\right\} + O(D) \qquad (4\text{-}56)$$

$$\hat{X}^{(2)}_{i\delta} = -\,\frac{w^{(us)}_{if}}{u^{(us)}_{if}}\,\hat{Z}^{(2)}_{i\delta} + O(D) \quad . \qquad (4\text{-}57)$$

A similar relationship to (4-53) holds for the pitching moment - this will be readily appreciated if one again considers the two-dimensional incompressible flow case*, *ie* we have

$$M^{(2)}_{if} = \frac{1}{2}\left\{\hat{M}^{(2)}_{i\dot{x}}\,u^{(us)}_{if} + M^{(2)}_{i\dot{z}}\,w^{(us)}_{if}\right\} + O(D) \quad . \qquad (4\text{-}58)$$

Finally, one would remark that one can allow, to some extent, for such things as viscous effects, by permitting infringements of the above relationships (4-54) to (4-58).

4.2.3 <u>In terms of the generalised coordinates</u>

From equations (3-7), (4-12) and (4-13):

$$\begin{bmatrix} u^{(s)}_{i} - u^{(us)}_{if} \\ w^{(s)}_{i} - w^{(us)}_{if} \\ q^{(s)}_{i} \\ \delta_i - \delta_{if} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} S_{\phi_{if}} \begin{bmatrix} A_{u_f} S^T_{\phi_{if}} Q_{\phi_{if}} F + kD & ID & A_{u_f} - A_{x_{if}} D \end{bmatrix} \\ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} Q_{\phi_{if}} FD & 0 & S_{\phi_{if}} D \end{bmatrix} \\ \begin{matrix} f^T & \qquad 0 & \qquad 0 \end{matrix} \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \\ x^{(c)}_1 \\ y^{(c)}_1 \\ z^{(c)}_1 \\ \phi \\ \theta \\ \psi \end{bmatrix}$$

...... (4-59)

* The pitching moment, about the centre of the circle from which a profile is generated by conformal transformation, is proportional to $(V_f + \delta V) \sin \{2(\alpha - \alpha_1)\}$.

(*cf* equations (3-14), (3-24) and (4-7)) and so, from (4-21), the overall forces on a strip, referred to the strip-fixed axes, are

$$
\begin{bmatrix} X_i^{(s)} \\ Y_i^{(s)} \\ Z_i^{(s)} \end{bmatrix} \approx \begin{bmatrix} X_{if}^{(2)} \\ 0 \\ Z_{if}^{(2)} \end{bmatrix} + \left\{ \begin{bmatrix} \hat{X}_{i\dot{x}}^{(2)} & 0 & \hat{X}_{i\dot{z}}^{(2)} \\ 0 & 0 & 0 \\ \hat{Z}_{i\dot{x}}^{(2)} & 0 & \hat{Z}_{i\dot{z}}^{(2)} \end{bmatrix} S_{\phi_{if}} \begin{bmatrix} A_{u_f} S_{\phi_{if}}^T Q_{\phi_{if}} F + KD & ID & A_{u_f} - A_{x_{if}} D \end{bmatrix} \right.
$$

$$
+ \begin{bmatrix} 0 & \hat{X}_{i\dot{\theta}}^{(2)} & 0 \\ 0 & 0 & 0 \\ 0 & \hat{Z}_{i\dot{\theta}}^{(2)} & 0 \end{bmatrix} \begin{bmatrix} Q_{\phi_{if}} FD & 0 & S_{\phi_{if}} D \end{bmatrix} + \left. \begin{bmatrix} \hat{X}_{i\delta}^{(2)} f^T & 0 & 0 \\ 0 & 0 & 0 \\ \hat{Z}_{i\delta}^{(2)} f^T & 0 & 0 \end{bmatrix} \right\} \times
$$

$$
\times \begin{bmatrix} q_1 \\ \vdots \\ q_n \\ x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \\ \phi \\ \theta \\ \psi \end{bmatrix} = \text{(say)} \begin{bmatrix} X_{if}^{(2)} \\ 0 \\ Z_{if}^{(2)} \end{bmatrix} + \begin{bmatrix} X_{iq}^{(s)} & X_{ix}^{(s)} & X_{i\phi}^{(s)} \\ 0 & 0 & 0 \\ Z_{iq}^{(2)} & Z_{ix}^{(s)} & Z_{i\phi}^{(s)} \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} . \qquad \text{...... (4-60)}
$$

Similarly the overall moments on the strip, about the strip reference point, referred to the strip-fixed axes, become, from (4-30)

$$
\begin{bmatrix} L_i^{(s)} \\ M_i^{(s)} \\ N_i^{(s)} \end{bmatrix} \approx \begin{bmatrix} 0 \\ M_{if}^{(2)} \\ 0 \end{bmatrix} + \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \hat{M}_{i\dot{x}}^{(2)} & 0 & \hat{M}_{i\dot{z}}^{(2)} \\ 0 & 0 & 0 \end{bmatrix} S_{\phi_{if}} \begin{bmatrix} A_{u_f} S^T_{\phi_{if}} Q_{\phi_{if}} F + KD & ID & A_{u_f} - A_{x_{if}} D \end{bmatrix} \right.
$$

$$
\left. + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \hat{M}_{i\dot{\theta}}^{(2)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_{\phi_{if}} FD & 0 & S_{\phi_{if}} D \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \hat{M}_{i\delta}^{(2)} f^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \times
$$

$$
\times \begin{bmatrix} q_1 \\ \vdots \\ q_n \\ x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \\ \phi \\ \theta \\ \psi \end{bmatrix} = \text{(say)} \begin{bmatrix} 0 \\ M_{if}^{(2)} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ M_{iq}^{(s)} & M_{ix}^{(s)} & M_{i\phi}^{(s)} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} . \qquad \text{...... (4-61)}
$$

From (4-36) we see that the hinge moment B_i has the same form as the pitching moment $M_i^{(s)}$ - all the M coefficients are merely replaced by B coefficients.

The final forms of the above expressions can of course be used even when one is not using two-dimensional strip theory - $X_{if}^{(2)}$, $Z_{if}^{(2)}$, $M_{if}^{(2)}$ (with the 2 replaced by a 3), and the row vectors $X_{iq}^{(2)}$ etc, being given appropriate three-dimensional values.

4.3 The overall forces and moments on the aircraft

The overall forces on a strip, referred to the constant-velocity axes, are, using the relationships (3-15) and (4-5) along with equation (4-60)

$$\begin{bmatrix} X_i^{(c)} \\ Y_i^{(c)} \\ Z_i^{(c)} \end{bmatrix} = S_\phi^T S_{\phi_i}^T \begin{bmatrix} X_i^{(s)} \\ Y_i^{(s)} \\ Z_i^{(s)} \end{bmatrix}$$

$$\approx S_{\phi_{if}}^T \left\{ \begin{bmatrix} X_i^{(s)} \\ Y_i^{(s)} \\ Z_i^{(s)} \end{bmatrix} - A_{X_{if}^{(2)}} \left(Q_{\phi_{if}} F \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + S_{\phi_{if}} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \right) \right\}$$

$$\approx S_{\phi_{if}}^T \left\{ \begin{bmatrix} X_{if}^{(2)} \\ 0 \\ Z_{if}^{(2)} \end{bmatrix} + \left(\begin{bmatrix} \hat{X}_{i\dot{x}}^{(2)} & 0 & \hat{X}_{i\dot{z}}^{(2)} \\ 0 & 0 & 0 \\ \hat{Z}_{i\dot{x}}^{(2)} & 0 & \hat{Z}_{i\dot{z}}^{(2)} \end{bmatrix} S_{\phi_{if}} \begin{bmatrix} A_{u_f} S_{\phi_{if}}^T Q_{\phi_{if}} F + KD & ID & A_{u_f} - A_{x_{if}} D \end{bmatrix} \right. \right.$$

$$+ \begin{bmatrix} 0 & Z_{if}^{(2)} + \hat{X}_{i\dot{\theta}}^{(2)} D & 0 \\ -Z_{if}^{(2)} & 0 & X_{if}^{(2)} \\ 0 & -X_{if}^{(2)} + \hat{Z}_{i\dot{\theta}}^{(2)} D & 0 \end{bmatrix} \begin{bmatrix} Q_{\phi_{if}} F & 0 & S_{\phi_{if}} \end{bmatrix}$$

$$\left. \left. + \begin{bmatrix} \hat{X}_{i\delta}^{(2)} f^T & 0 & 0 \\ 0 & 0 & 0 \\ \hat{Z}_{i\delta}^{(2)} f^T & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} q_1 \\ \vdots \\ q_n \\ x_1^{(c)} \\ \vdots \\ \phi \\ \vdots \end{bmatrix} \right\}$$

$$= S_{\phi_{if}}^T \left\{ \begin{bmatrix} X_{if}^{(2)} \\ 0 \\ Z_{if}^{(2)} \end{bmatrix} + \left(\begin{bmatrix} X_{iq}^{(s)} & X_{ix}^{(s)} & X_{i\phi}^{(s)} \\ 0 & 0 & 0 \\ Z_{iq}^{(s)} & Z_{ix}^{(s)} & Z_{i\phi}^{(s)} \end{bmatrix} - A_{X_{if}^{(2)}} \begin{bmatrix} Q_{\phi_{if}} F & 0 & S_{\phi_{if}} \end{bmatrix} \right) \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \right\} .$$

...... (4-62)

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Similarly the overall moments on a strip about the strip reference point*, referred to the constant-velocity axes, are

$$
\begin{bmatrix} L_i^{(c)} \\ M_i^{(c)} \\ N_i^{(c)} \end{bmatrix} \approx S^T_{\phi_{if}} \left\{ \begin{bmatrix} 0 \\ M_{if}^{(2)} \\ 0 \end{bmatrix} + \left(\begin{bmatrix} 0 & 0 & 0 \\ \hat{M}_{i\dot{x}}^{(2)} & 0 & \hat{M}_{i\dot{z}}^{(2)} \\ 0 & 0 & 0 \end{bmatrix} S_{\phi_{if}} \begin{bmatrix} A_{u_f} S^T_{\phi_{if}} Q_{\phi_{if}} F + KD & ID & A_{u_f} - A_{x_{if}} D \end{bmatrix} \right. \right.
$$

$$
+ \begin{bmatrix} 0 & 0 & -M_{if}^{(2)} \\ 0 & \hat{M}_{i\dot{\theta}}^{(2)} D & 0 \\ M_{if}^{(2)} & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_{\phi_{if}} F & 0 & S_{\phi_{if}} \end{bmatrix}
$$

$$
\left. \left. + \begin{bmatrix} 0 & 0 & 0 \\ \hat{M}_{i\delta}^{(2)} f^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} q_1 \\ \vdots \\ q_n \\ x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \\ \phi \\ \theta \\ \psi \end{bmatrix} \right\}
$$

$$
= S^T_{\phi_{if}} \left\{ \begin{bmatrix} 0 \\ M_{if}^{(2)} \\ 0 \end{bmatrix} + \left(\begin{bmatrix} 0 & 0 & 0 \\ M_{iq}^{(s)} & M_{ix}^{(s)} & M_{i\phi}^{(s)} \\ 0 & 0 & 0 \end{bmatrix} - A_{L_{if}^{(2)}} \begin{bmatrix} Q_{\phi_{if}} F & 0 & S_{\phi_{if}} \end{bmatrix} \right) \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \right\}
\qquad \text{......} \quad (4\text{-}63)
$$

* In accordance with our practice the symbols for the moments should also have a subscript s to show that the moments are about the origin of the strip-fixed axes but this has been omitted to avoid confusion with the subsequent use of a subscript s to denote structural.

The overall forces on the aircraft, referred to the same axes, are given simply by summing (4-62) for all strips, viz

$$\begin{bmatrix} X^{(c)} \\ Y^{(c)} \\ Z^{(c)} \end{bmatrix} = \sum_i \begin{bmatrix} X_i^{(c)} \\ Y_i^{(c)} \\ Z_i^{(c)} \end{bmatrix} \approx \text{(say)} \begin{bmatrix} X_f \\ Y_f \\ Z_f \end{bmatrix} + \begin{bmatrix} X_q^{(c)} & X_x^{(c)} & X_\phi^{(c)} \\ Y_q^{(c)} & Y_x^{(c)} & Y_\phi^{(c)} \\ Z_q^{(c)} & Z_x^{(c)} & Z_\phi^{(c)} \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \tag{4-64}$$

where

$$\begin{bmatrix} X_f \\ Y_f \\ Z_f \end{bmatrix} = \sum_i S_{\phi_{if}}^T \begin{bmatrix} X_{if}^{(2)} \\ 0 \\ Z_{if}^{(2)} \end{bmatrix} \tag{4-65}$$

$$\begin{bmatrix} X_q^{(c)} \\ Y_q^{(c)} \\ Z_q^{(c)} \end{bmatrix} = \sum_i S_{\phi_{if}}^T \left(\begin{bmatrix} X_{iq}^{(s)} \\ 0 \\ Z_{iq}^{(s)} \end{bmatrix} - A_{X_{if}^{(2)}} Q_{\phi_{if}} F \right) \tag{4-66}$$

$$\begin{bmatrix} X_x^{(c)} \\ Y_x^{(c)} \\ Z_x^{(c)} \end{bmatrix} = \sum_i S_{\phi_{if}}^T \begin{bmatrix} X_{ix}^{(s)} \\ 0 \\ Z_{ix}^{(s)} \end{bmatrix} \tag{4-67}$$

$$\begin{bmatrix} X_\phi^{(c)} \\ Y_\phi^{(c)} \\ Z_\phi^{(c)} \end{bmatrix} = \sum_i S_{\phi_{if}}^T \left(\begin{bmatrix} X_{i\phi}^{(s)} \\ 0 \\ Z_{i\phi}^{(s)} \end{bmatrix} - A_{X_{if}^{(2)}} S_{\phi_{if}} \right) . \tag{4-68}$$

The overall moments on the aircraft, about the origin of the constant-velocity axes, and referred to the same axes, are

$$\begin{bmatrix} L_c^{(c)} \\ M_c^{(c)} \\ N_c^{(c)} \end{bmatrix} = \sum_i \left\{ \begin{bmatrix} L_i^{(c)} \\ M_i^{(c)} \\ N_i^{(c)} \end{bmatrix} + A_{x_i^{(c)}} \begin{bmatrix} X_i^{(c)} \\ Y_i^{(c)} \\ Z_i^{(c)} \end{bmatrix} \right\} \qquad (4\text{-}69)$$

where
$$\begin{bmatrix} x_i^{(c)} \\ y_i^{(c)} \\ z_i^{(c)} \end{bmatrix} = \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix} + S_\phi^T \begin{bmatrix} x_i^{(n)} \\ y_i^{(n)} \\ z_i^{(n)} \end{bmatrix}$$

$$\approx \begin{bmatrix} x_{if} \\ y_{if} \\ z_{if} \end{bmatrix} - A_{x_{if}} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} + K \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix} \qquad (4\text{-}70)$$

(*cf* equation (4-38)), and so

$$A_{x_i^{(c)}} \begin{bmatrix} X_i^{(c)} \\ Y_i^{(c)} \\ Z_i^{(c)} \end{bmatrix} \approx A_{x_{if}} \begin{bmatrix} X_i^{(c)} \\ Y_i^{(c)} \\ Z_i^{(c)} \end{bmatrix} + A_{X_{if}^{(c)}} \left\{ A_{x_{if}} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} - K \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} - \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix} \right\}$$

$$= A_{x_{if}} \begin{bmatrix} X_i^{(c)} \\ Y_i^{(c)} \\ Z_i^{(c)} \end{bmatrix} + S_{\phi_{if}}^T A_{X_{if}^{(2)}} S_{\phi_{if}} \left\{ A_{x_{if}} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} - K \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} - \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix} \right\} .$$

...... (4-71)

Thus we have

$$\begin{bmatrix} L_c^{(c)} \\ M_c^{(c)} \\ N_c^{(c)} \end{bmatrix} = \begin{bmatrix} L_f \\ M_f \\ N_f \end{bmatrix} + \begin{bmatrix} L_{cq}^{(c)} & L_{cx}^{(c)} & L_{c\phi}^{(c)} \\ M_{cq}^{(c)} & M_{cx}^{(c)} & M_{c\phi}^{(c)} \\ N_{cq}^{(c)} & N_{cx}^{(c)} & N_{c\phi}^{(c)} \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \\ x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \\ \phi \\ \theta \\ \psi \end{bmatrix} \tag{4-72}$$

where

$$\begin{bmatrix} L_f \\ M_f \\ N_f \end{bmatrix} = \sum_i \left\{ S_{\phi_{if}}^T \begin{bmatrix} 0 \\ M_{if}^{(2)} \\ 0 \end{bmatrix} + A_{x_{if}} S_{\phi_{if}}^T \begin{bmatrix} X_{if}^{(2)} \\ 0 \\ Z_{if}^{(2)} \end{bmatrix} \right\} \tag{4-73}$$

$$
\begin{aligned}
\begin{bmatrix} L^{(c)}_{cq} \\ M^{(c)}_{cq} \\ N^{(c)}_{cq} \end{bmatrix}
&= \sum_i \Bigg(S^T_{\phi_{if}} \Bigg\{ - A_{X^{(2)}_{if}} S_{\phi_{if}} K + \begin{bmatrix} 0 & 0 & 0 \\ \hat{M}^{(2)}_{i\dot{x}} & 0 & \hat{M}^{(2)}_{i\dot{z}} \\ 0 & 0 & 0 \end{bmatrix} S_{\phi_{if}} \left(A_{u_f} S^T_{\phi_{if}} Q_{\phi_{if}} F + KD \right) \\
&\qquad + \begin{bmatrix} 0 & 0 & -M^{(2)}_{if} \\ 0 & \hat{M}^{(2)}_{i\dot{\theta}} D & 0 \\ M^{(2)}_{if} & 0 & 0 \end{bmatrix} Q_{\phi_{if}} F + \begin{bmatrix} 0 \\ \hat{M}^{(2)}_{i\delta} f^T \\ 0 \end{bmatrix} \Bigg\} \\
&\qquad + A_{x_{if}} S^T_{\phi_{if}} \Bigg\{ \begin{bmatrix} \hat{X}^{(2)}_{i\dot{x}} & 0 & \hat{X}^{(2)}_{i\dot{z}} \\ 0 & 0 & 0 \\ \hat{Z}^{(2)}_{i\dot{x}} & 0 & \hat{Z}^{(2)}_{i\dot{z}} \end{bmatrix} S_{\phi_{if}} \left(A_{u_f} S^T_{\phi_{if}} Q_{\phi_{if}} F + KD \right) \\
&\qquad + \begin{bmatrix} 0 & Z^{(2)}_{if} + \hat{X}^{(2)}_{i\dot{\theta}} D & 0 \\ -Z^{(2)}_{if} & 0 & X^{(2)}_{if} \\ 0 & -X^{(2)}_{if} + \hat{Z}^{(2)}_{i\dot{\theta}} D & 0 \end{bmatrix} Q_{\phi_{if}} F + \begin{bmatrix} \hat{X}^{(2)}_{i\delta} f^T \\ 0 \\ \hat{Z}^{(2)}_{i\delta} f^T \end{bmatrix} \Bigg\} \Bigg) \\
&= \sum_i \Bigg(S^T_{\phi_{if}} \Bigg\{ \begin{bmatrix} 0 \\ M^{(s)}_{iq} \\ 0 \end{bmatrix} - A_{L^{(2)}_{if}} Q_{\phi_{if}} F - A_{X^{(2)}_{if}} S_{\phi_{if}} K \Bigg\} \\
&\qquad + A_{x_{if}} S^T_{\phi_{if}} \Bigg\{ \begin{bmatrix} X^{(s)}_{iq} \\ 0 \\ Z^{(s)}_{iq} \end{bmatrix} - A_{X^{(2)}_{if}} Q_{\phi_{if}} F \Bigg\} \Bigg) \qquad (4\text{-}74)
\end{aligned}
$$

$$\begin{bmatrix} L_{cx}^{(c)} \\ M_{cx}^{(c)} \\ N_{cx}^{(c)} \end{bmatrix} = \sum_i \left(\left\{ S_{\phi_{if}}^T \begin{bmatrix} 0 & 0 & 0 \\ \hat{M}_{\dot{i}x}^{(2)} & 0 & \hat{M}_{\dot{i}z}^{(2)} \\ 0 & 0 & 0 \end{bmatrix} + A_{x_{if}} S_{\phi_{if}}^T \begin{bmatrix} \hat{X}_{\dot{i}x}^{(2)} & 0 & \hat{X}_{\dot{i}z}^{(2)} \\ 0 & 0 & 0 \\ \hat{Z}_{\dot{i}x}^{(2)} & 0 & \hat{Z}_{\dot{i}z}^{(2)} \end{bmatrix} \right\} S_{\phi_{if}} D \right.$$

$$\left. - S_{\phi_{if}}^T A_{X_{if}^{(2)}} S_{\phi_{if}} \right)$$

$$= \sum_i S_{\phi_{if}}^T \left(\begin{bmatrix} 0 \\ M_{\dot{i}x}^{(s)} \\ 0 \end{bmatrix} + A_{x_{if}} S_{\phi_{if}}^T \begin{bmatrix} X_{\dot{i}x}^{(s)} \\ 0 \\ Z_{\dot{i}x}^{(s)} \end{bmatrix} - S_{\phi_{if}}^T A_{X_{if}^{(2)}} S_{\phi_{if}} \right) \tag{4-75}$$

and

$$\begin{bmatrix} L_{c\phi}^{(c)} \\ M_{c\phi}^{(c)} \\ N_{c\phi}^{(c)} \end{bmatrix} = \sum_i \left(S_{\phi_{if}}^T \left\{ A_{X_{if}^{(2)}} S_{\phi_{if}} A_{x_{if}} + \begin{bmatrix} 0 & 0 & 0 \\ \hat{M}_{\dot{i}x}^{(2)} & 0 & \hat{M}_{\dot{i}z}^{(2)} \\ 0 & 0 & 0 \end{bmatrix} S_{\phi_{if}} \left(A_{u_f} - A_{x_{if}} D \right) \right. \right.$$

$$\left. + \begin{bmatrix} 0 & 0 & -M_{if}^{(2)} \\ 0 & \hat{M}_{\dot{i}\theta}^{(2)} D & 0 \\ M_{if}^{(2)} & 0 & 0 \end{bmatrix} S_{\phi_{if}} \right\}$$

$$+ A_{x_{if}} S_{\phi_{if}}^T \left\{ \begin{bmatrix} \hat{X}_{\dot{i}x}^{(2)} & 0 & \hat{X}_{\dot{i}z}^{(2)} \\ 0 & 0 & 0 \\ \hat{Z}_{\dot{i}x}^{(2)} & 0 & \hat{Z}_{\dot{i}z}^{(2)} \end{bmatrix} S_{\phi_{if}} \left(A_{u_f} - A_{x_{if}} D \right) \right.$$

$$\left. \left. + \begin{bmatrix} 0 & Z_{if}^{(2)} + \hat{X}_{\dot{i}\theta}^{(2)} D & 0 \\ -Z_{if}^{(2)} & 0 & X_{if}^{(2)} \\ 0 & -X_{if}^{(2)} + \hat{Z}_{\dot{i}\theta}^{(2)} D & 0 \end{bmatrix} S_{\phi_{if}} \right\} \right) =$$

$$= \sum_i \left(S^T_{\phi_{if}} \left\{ \begin{bmatrix} 0 \\ M^{(s)}_{i\phi} \\ 0 \end{bmatrix} - A_{L^{(2)}_{if}} S_{\phi_{if}} + A_{X^{(2)}_{if}} S_{\phi_{if}} A_{x_{if}} \right\} \right.$$

$$\left. + A_{x_{if}} S^T_{\phi_{if}} \left\{ \begin{bmatrix} X^{(s)}_{i\phi} \\ 0 \\ Z^{(s)}_{i\phi} \end{bmatrix} - A_{X^{(2)}_{if}} S_{\phi_{if}} \right\} \right) . \qquad (4\text{-}76)$$

4.4 The generalised forces

The generalised aerodynamic forces could be determined using the expressions derived in Ref 1, and generalised in Appendix A, which are in terms of the overall and local forces referred to the constant velocity axes (*cf* equations (A-18) and (A-19)). However, for the particular semi-rigid and aerodynamic models being considered in this paper it is more convenient to proceed rather differently.

The linear velocity of the strip reference point, and the angular velocity of the strip-fixed axes, both resolved along the strip-fixed axes, are, for the ith strip, respectively (*cf* equations (4-1), (4-3), (4-41) and (4-42))

$$\begin{bmatrix} u^{(s)}_i \\ v^{(s)}_i \\ w^{(s)}_i \end{bmatrix} = S_{\phi_i} \left\{ \begin{bmatrix} \dot{x}^{(n)}_i \\ \dot{y}^{(n)}_i \\ \dot{z}^{(n)}_i \end{bmatrix} + S\dot{S}^T \begin{bmatrix} x^{(n)}_i \\ y^{(n)}_i \\ z^{(n)}_i \end{bmatrix} + S \begin{bmatrix} \dot{x}^{(c)}_1 \\ \dot{y}^{(c)}_1 \\ \dot{z}^{(c)}_1 \end{bmatrix} + S \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} \right\} \qquad (4\text{-}77)$$

and, by a well known relationship (*cf* for example, Ref 4)

$$\begin{bmatrix} p^{(s)}_i \\ q^{(s)}_i \\ r^{(s)}_i \end{bmatrix} = Q_{\phi_{ui}} \begin{bmatrix} \dot{\phi}_{ui} \\ \dot{\theta}_{ui} \\ \dot{\psi}_{ui} \end{bmatrix} \qquad (4\text{-}78)$$

where the rotations ϕ_{ui} etc satisfy equation (4-39), $Q_{\phi_{ui}}$ is defined by equation (3-14), and it can be shown that, using (4-39)

where
$$Q_j^{(i)} = \left\{ \frac{\partial}{\partial q_j} \begin{bmatrix} x_{ci}^{(s)} \\ y_{ci}^{(s)} \\ z_{ci}^{(s)} \end{bmatrix} - A_{x_{ci}^{(s)}} Q_{\phi_{ui}} \frac{\partial}{\partial q_j} \begin{bmatrix} \phi_{ui} \\ \theta_{ui} \\ \psi_{ui} \end{bmatrix} \right\}^T \begin{bmatrix} X_i^{(s)} \\ Y_i^{(s)} \\ Z_i^{(s)} \end{bmatrix}$$

$$+ \left\{ Q_{\phi_{ui}} \frac{\partial}{\partial q_j} \begin{bmatrix} \phi_{ui} \\ \theta_{ui} \\ \psi_{ui} \end{bmatrix} \right\}^T \begin{bmatrix} L_i^{(s)} \\ M_i^{(s)} \\ N_i^{(s)} \end{bmatrix} + B_i \frac{\partial \delta_i}{\partial q_j} \quad . \qquad (4\text{-}82)$$

By analogy with (4-78) and (4-79) we have, for any column vector $\{x\ y\ z\}$, and indeed as regards this equation the other vector $\{\phi_{ui}\ \theta_{ui}\ \psi_{ui}\}$ is also arbitrary,

$$- A_x Q_{\phi_{ui}} \frac{\partial}{\partial q_j} \begin{bmatrix} \phi_{ui} \\ \theta_{ui} \\ \psi_{ui} \end{bmatrix} = S_{\phi_{ui}} \frac{\partial S^T_{\phi_{ui}}}{\partial q_j} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-\partial S_{\phi_{ui}}}{\partial q_j} S^T_{\phi_{ui}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad . \qquad (4\text{-}83)$$

Substitution therefore verifies that, for ϕ_{ui} etc, satisfying (4-38), we have*

$$Q_{\phi_{ui}} \frac{\partial}{\partial q_j} \begin{bmatrix} \phi_{ui} \\ \theta_{ui} \\ \psi_{ui} \end{bmatrix} = Q_{\phi_i} \frac{\partial}{\partial q_j} \begin{bmatrix} \phi_i \\ \theta_i \\ \psi_i \end{bmatrix} + S_{\phi_i} Q_{\phi} \frac{\partial}{\partial q_j} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \quad . \qquad (4\text{-}84)$$

Bearing in mind equation (3-15), and the fact that

---

* This expression can be deduced considering the angular velocity of the strip-fixed axes as the sum of the angular velocity of the no-deformation-body-fixed axes, and the angular velocity, relative to these axes, of the strip-fixed axes.

$$A_{p_i^{(s)}} = S_{\phi_{ui}} \dot{S}^T_{\phi_{ui}}$$

$$= S_{\phi_i} \dot{S}^T_{\phi_i} + S_{\phi_i} S \dot{S}^T S^T_{\phi_i} \quad . \tag{4-79}$$

The position of the strip reference point, relative to the origin of the constant-velocity axes, resolved along the strip-fixed axes, is given by, say

$$\begin{bmatrix} x_{ci}^{(s)} \\ y_{ci}^{(s)} \\ z_{ci}^{(s)} \end{bmatrix} = S_{\phi_i} \left\{ S \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix} + \begin{bmatrix} x_i^{(n)} \\ y_i^{(n)} \\ z_i^{(n)} \end{bmatrix} \right\} \tag{4-80}$$

and so

$$\begin{bmatrix} u_i^{(s)} \\ v_i^{(s)} \\ w_i^{(s)} \end{bmatrix} = \begin{bmatrix} \dot{x}_{ci}^{(s)} \\ \dot{y}_{ci}^{(s)} \\ \dot{z}_{ci}^{(s)} \end{bmatrix} - A_{x_{ci}^{(s)}} \begin{bmatrix} p_i^{(s)} \\ q_i^{(s)} \\ r_i^{(s)} \end{bmatrix} + S_{\phi_i} S \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} \quad . \tag{4-81}$$

Consequently, from the work done in time δt , going to the limit $\delta t = 0$, we find that the virtual work done by the forces on the strip in small displacements $\delta q_1, \ldots, \delta q_{n+6}$ is

$$\sum_{j=1}^{n+6} Q_j^{(i)} \delta q_j$$

$$Q_{\phi_i} \approx Q_{\phi_{if}} - \cos\theta_{if}\left(Q^T_{\phi_{if}}\right)^{-1} J_{\phi_i - \phi_{if}}$$

$$= Q_{\phi_{if}} - \cos\theta_{if}\left(Q^T_{\phi_{if}}\right)^{-1} J_{\left(F\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}\right)} \tag{4-85}$$

where J is defined below (equation (4-88)), we see that, using (4-83)

$$\begin{bmatrix} \frac{\partial}{\partial q_1}\begin{bmatrix} x^{(s)}_{ci} & y^{(s)}_{ci} & z^{(s)}_{ci} \end{bmatrix} \\ \frac{\partial}{\partial q_2}\begin{bmatrix} x^{(s)}_{ci} & y^{(s)}_{ci} & z^{(s)}_{ci} \end{bmatrix} \\ \vdots \\ \frac{\partial}{\partial q_{n+6}}\begin{bmatrix} x^{(s)}_{ci} & y^{(s)}_{ci} & z^{(s)}_{ci} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} K^T S^T_{\phi_i} - F^T Q^T_{\phi_i} A_{x^{(s)}_{ci}} \\ S^T S^T_{\phi_i} \\ - Q^T_{\phi} A_{\left(S\begin{bmatrix} x^{(c)}_1 \\ y^{(c)}_1 \\ z^{(c)}_1 \end{bmatrix}\right)} S^T_{\phi_i} \end{bmatrix}$$

$$\approx \begin{bmatrix} K^T S^T_{\phi_{if}} - F^T Q^T_{\phi_{if}} S_{\phi_{if}} A_{x_{if}} S^T_{\phi_{if}} \\ S^T_{\phi_{if}} \\ 0 \end{bmatrix}$$

$$+ \begin{bmatrix} K^T S^T_{\phi_{if}} A_{\alpha_i} - F^T Q^T_{\phi_{if}}\left\{A_\tau + S_{\phi_{if}} A_{x_{if}} S^T_{\phi_{if}} A_{\alpha_i} - A_{\alpha_i} S_{\phi_{if}} A_{x_{if}} S^T_{\phi_{if}}\right\} \\ + \cos\theta_{if} F^T J^T_{\phi_i - \phi_{if}} Q^{-1}_{\phi_{if}} S_{\phi_{if}} A_{x_{if}} S^T_{\phi_{if}} \\ A_\phi S^T_{\phi_{if}} + S^T_{\phi_{if}} A_{\alpha_i} \\ - A_{x^{(c)}_1} S^T_{\phi_{if}} \end{bmatrix} \tag{4-86}$$

where α_i etc is given by (3-13), $\phi_i - \phi_{if}$ by (3-3),

$$\begin{bmatrix} \tau \\ \nu \\ \pi \end{bmatrix} = S_{\phi_{if}} \left\{ \begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix} + K \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right\} \tag{4-87}$$

and

$$J_\phi = \begin{bmatrix} 0 & 0 & \theta \\ 0 & 0 & -\phi \\ 0 & \phi & 0 \end{bmatrix} . \tag{4-88}$$

Similarly, using (4-84), we have

$$\begin{bmatrix} \frac{\partial}{\partial q_1} \left[\phi_{ui}\theta_{ui}\psi_{ui}\right] Q^T_{\phi_{ui}} \\ \frac{\partial}{\partial q_2} \left[\phi_{ui}\theta_{ui}\psi_{ui}\right] Q^T_{\phi_{ui}} \\ \vdots \\ \frac{\partial}{\partial q_{n+6}} \left[\phi_{ui}\theta_{ui}\psi_{ui}\right] Q^T_{\phi_{ui}} \end{bmatrix} = \begin{bmatrix} F^T Q^T_{\phi_i} \\ 0 \\ Q^T_\phi S^T_{\phi_i} \end{bmatrix}$$

$$\approx \begin{bmatrix} F^T Q^T_{\phi_{if}} \\ 0 \\ S^T_{\phi_{if}} \end{bmatrix} + \begin{bmatrix} -\cos\theta_{if} F^T J^T_{\phi_i - \phi_{if}} Q^{-1}_{\phi_{if}} \\ 0 \\ -J^T_\phi S^T_{\phi_{if}} + S^T_{\phi_{if}} A_{\alpha_i} \end{bmatrix} . \tag{4-89}$$

Furthermore, since, from (4-80), (3-2) and (3-15),

$$A_{x_{ci}^{(s)}} = S_{\phi_i}\left\{ SA_{x_1^{(c)}} S^T + A_{x_{if}} + A\left(K\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}\right)\right\} S_{\phi_i}^T$$

$$\approx S_{\phi_{if}} A_{x_{if}} S_{\phi_{if}}^T + \left\{ S_{\phi_{if}} A\left(K\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}\right) S_{\phi_{if}}^T - A_{\alpha_i} S_{\phi_{if}} A_{x_{if}} S_{\phi_{if}}^T + S_{\phi_{if}} A_{x_1^{(c)}} S_{\phi_{if}}^T \right.$$

$$\left. + S_{\phi_{if}} A_{x_{if}} S_{\phi_{if}}^T A_{\alpha_i} \right\} \qquad \text{...... (4-90)}$$

we have

$$\begin{bmatrix} \frac{\partial}{\partial q_1}\begin{bmatrix} x_{ci}^{(s)} & y_{ci}^{(s)} & z_{ci}^{(s)} \end{bmatrix} + \frac{\partial}{\partial q_1}\begin{bmatrix} \phi_{ui} & \theta_{ui} & \psi_{ui} \end{bmatrix} Q_{\phi_{ui}}^T A_{x_{ci}^{(s)}} \\ \frac{\partial}{\partial q_2}\begin{bmatrix} x_{ci}^{(s)} & y_{ci}^{(s)} & z_{ci}^{(s)} \end{bmatrix} + \frac{\partial}{\partial q_2}\begin{bmatrix} \phi_{ui} & \theta_{ui} & \psi_{ui} \end{bmatrix} Q_{\phi_{ui}}^T A_{x_{ci}^{(s)}} \\ \vdots \qquad\qquad\qquad \vdots \\ \frac{\partial}{\partial q_{n+6}}\begin{bmatrix} x_{ci}^{(s)} & y_{ci}^{(s)} & z_{ci}^{(s)} \end{bmatrix} + \frac{\partial}{\partial q_{n+6}}\begin{bmatrix} \phi_{ui} & \theta_{ui} & \psi_{ui} \end{bmatrix} Q_{\phi_{ui}}^T A_{x_{ci}^{(s)}} \end{bmatrix}$$

$$= \begin{bmatrix} K^T S_{\phi_i}^T \\ S^T S_{\phi_i}^T \\ Q_\phi^T\left(S_{\phi_i}^T A_{x_{ci}^{(s)}} - A\left(S\begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix}\right) S_{\phi_i}^T\right) \end{bmatrix} \approx \begin{bmatrix} K^T S_{\phi_{if}}^T \\ S_{\phi_{if}}^T \\ A_{x_{if}} S_{\phi_{if}}^T \end{bmatrix} + \begin{bmatrix} K^T S_{\phi_{if}}^T A_{\alpha_i} \\ A_\phi S_{\phi_{if}}^T + S_{\phi_{if}}^T A_{\alpha_i} \\ A_{x_{if}} S_{\phi_{if}}^T A_{\alpha_i} + A\left(K\begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}\right) S_{\phi_{if}}^T - J_\phi^T A_{x_{if}} S_{\phi_{if}}^T \end{bmatrix}$$

...... (4-91)

The forces and moments which appear in the strip contribution to the generalised force, equation (4-82), are given by equations (4-60) and (4-61), and so using these equations, along with (4-89) and (4-91), we have, writing

$$Q_j = \sum_i Q_j^{(i)} \approx Q_{jf} + \sum_{k=1}^{n+6} Q_{jk} q_k \tag{4-92}$$

$$\begin{bmatrix} Q_{1f} \\ \vdots \\ Q_{n+6,f} \end{bmatrix} = \sum_i \left\{ \begin{bmatrix} K^T S^T_{\phi_{if}} \\ S^T_{\phi_{if}} \\ A_{x_{if}} S^T_{\phi_{if}} \end{bmatrix} \begin{bmatrix} X^{(2)}_{if} \\ 0 \\ Z^{(2)}_{if} \end{bmatrix} + \begin{bmatrix} F^T Q^T_{\phi_{if}} \\ 0 \\ S^T_{\phi_{if}} \end{bmatrix} \begin{bmatrix} 0 \\ M^{(2)}_{if} \\ 0 \end{bmatrix} + \begin{bmatrix} fB^{(2)}_{if} \\ 0 \\ 0 \end{bmatrix} \right\}$$

$=$ (see equations (4-64) and (4-73)):

$$\begin{bmatrix} \sum_i \left\{ K^T S^T_{\phi_{if}} \begin{bmatrix} X^{(2)}_{if} \\ 0 \\ Z^{(2)}_{if} \end{bmatrix} + F^T Q^T_{\phi_{if}} \begin{bmatrix} 0 \\ M^{(2)}_{if} \\ 0 \end{bmatrix} + fB^{(2)}_{if} \right\} \\ \begin{bmatrix} X_f \\ Y_f \\ Z_f \end{bmatrix} \\ \begin{bmatrix} L_f \\ M_f \\ N_f \end{bmatrix} \end{bmatrix} \tag{4-93}$$

and

$$
[Q_{jk}] = \sum_i \left\{ \begin{bmatrix} K^T S^T_{\phi_{if}} \\ S^T_{\phi_{if}} \\ A_{x_{if}} S^T_{\phi_{if}} \end{bmatrix} \begin{bmatrix} X^{(s)}_{iq} & X^{(s)}_{ix} & X^{(s)}_{i\phi} \\ 0 & 0 & 0 \\ Z^{(s)}_{iq} & Z^{(s)}_{ix} & Z^{(s)}_{i\phi} \end{bmatrix} + \begin{bmatrix} F^T Q^T_{\phi_{if}} \\ 0 \\ S^T_{\phi_{if}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ M^{(s)}_{iq} & M^{(s)}_{ix} & M^{(s)}_{i\phi} \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} B^{(s)}_{iq} & B^{(s)}_{ix} & B^{(s)}_{i\phi} \end{bmatrix} \right.
$$

$$
\left. + \begin{bmatrix} -K^T S^T_{\phi_{if}} A_{X^{(2)}_{if}} Q_{\phi_{if}} F - \cos\theta_{if} F^T \mathcal{L}_{\left(Q^{-1}_{\phi_{if}} \begin{bmatrix} 0 \\ M^{(s)}_{if} \\ 0 \end{bmatrix} \right)} F & 0 & 0 \\ -S^T_{\phi_{if}} A_{X^{(2)}_{if}} Q_{\phi_{if}} F & 0 & -S^T_{\phi_{if}} A_{X^{(2)}_{if}} S_{\phi_{if}} \\ -S^T_{\phi_{if}} A_{X^{(2)}_{if}} S_{\phi_{if}} K - A_{x_{if}} S^T_{\phi_{if}} A_{X^{(2)}_{if}} Q_{\phi_{if}} F - S^T_{\phi_{if}} A_{L^{(2)}_{if}} Q_{\phi_{if}} F & 0 & -\mathcal{L}_{\left(A_{x_{if}} S^T_{\phi_{if}} \begin{bmatrix} X^{(2)}_{if} \\ 0 \\ Z^{(2)}_{if} \end{bmatrix} + S^T_{\phi_{if}} \begin{bmatrix} 0 \\ M^{(2)}_{if} \\ 0 \end{bmatrix} \right)} \end{bmatrix} \right\}
$$

...... (4-94)

where $\mathcal{L}_\phi$ is the lower triangular matrix formed from the elements of $\{\phi \;\; \theta \;\; \psi\}$ viz

$$
\mathcal{L}_\phi = \begin{bmatrix} 0 & 0 & 0 \\ \psi & 0 & 0 \\ -\theta & \phi & 0 \end{bmatrix} \qquad (4\text{-}95)
$$

and so

$$
\mathcal{L}_\phi - \mathcal{L}^T_\phi = A_\phi \quad . \qquad (4\text{-}96)
$$

In addition $Y^{(2)}_{if}$, $L^{(2)}_{if}$, $N^{(2)}_{if}$ are each zero and so

$$A_{X_{if}^{(2)}} = \begin{bmatrix} 0 & -Z_{if}^{(2)} & 0 \\ Z_{if}^{(2)} & 0 & -X_{if}^{(2)} \\ 0 & X_{if}^{(2)} & 0 \end{bmatrix} \tag{4-97}$$

and

$$A_{L_{if}^{(2)}} = \begin{bmatrix} 0 & 0 & M_{if}^{(2)} \\ 0 & 0 & 0 \\ -M_{if}^{(2)} & 0 & 0 \end{bmatrix} . \tag{4-98}$$

The second and third rows of submatrices in the matrix $[Q_{ij}]$ are, as expected (*cf* Ref 1, equations (74) to (78) and Table 3), closely related to the coefficients in the expressions (see equations (4-64), (4-72), (4-74) to (4-76) and (4-62)) for the overall forces and moments. Making use of the general relationships (4-96), (4-5), and

$$\begin{aligned} A\left(A_x S_\phi^T \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}\right) &= A_x A\left(S_\phi^T \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}\right) - A\left(S_\phi^T \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}\right) A_x \\ &= A_x S^T A_X S_\phi - S_\phi^T A_X S A_x \end{aligned} \tag{4-99}$$

we find that

$$\begin{bmatrix} Q_{n+1,1} & \cdots & Q_{n+1,n+6} \\ \cdots & \cdots & \cdots \\ Q_{n+6,1} & \cdots & Q_{n+6,n+6} \end{bmatrix} = \left[\begin{matrix} \begin{bmatrix} X_q^{(c)} \\ Y_q^{(c)} \\ Z_q^{(c)} \end{bmatrix} & \begin{bmatrix} X_x^{(c)} \\ Y_x^{(c)} \\ Z_x^{(c)} \end{bmatrix} & \begin{bmatrix} X_\phi^{(c)} \\ Y_\phi^{(c)} \\ Z_\phi^{(c)} \end{bmatrix} \\ \begin{bmatrix} L_{cq}^{(c)} \\ M_{cq}^{(c)} \\ N_{cq}^{(c)} \end{bmatrix} & \begin{bmatrix} L_{cx}^{(c)} \\ M_{cx}^{(c)} \\ N_{cx}^{(c)} \end{bmatrix} + \sum_i S_{\phi_{if}}^T A_{X_{if}^{(2)}} S_{\phi_{if}} & \begin{bmatrix} L_{c\phi}^{(c)} \\ M_{c\phi}^{(c)} \\ N_{c\phi}^{(c)} \end{bmatrix} - \sum_i \mathcal{L}\left(A_{x_{if}} S_{\phi_{if}}^T \begin{bmatrix} X_{if}^{(2)} \\ 0 \\ Z_{if}^{(2)} \end{bmatrix} + S_{\phi_{if}}^T \begin{bmatrix} 0 \\ M_{if}^{(2)} \\ 0 \end{bmatrix} \right) \end{matrix}\right]$$

$$= \text{(see Equation (4-93))}$$

$$\begin{bmatrix} \begin{bmatrix} X_q^{(c)} & X_x^{(c)} & X_\phi^{(c)} \\ Y_q^{(c)} & Y_x^{(c)} & Y_\phi^{(c)} \\ Z_q^{(c)} & Z_x^{(c)} & Z_\phi^{(c)} \end{bmatrix} \\ \begin{bmatrix} L_{cq}^{(c)} & L_{cx}^{(c)} & L_{c\phi}^{(c)} \\ M_{cq}^{(c)} & M_{cx}^{(c)} & M_{c\phi}^{(c)} \\ N_{cq}^{(c)} & N_{cx}^{(c)} & N_{c\phi}^{(c)} \end{bmatrix} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{X_f} & -\mathcal{L}_{L_f}^T \end{bmatrix} \quad . \qquad (4\text{-}100)$$

These expressions for the aerodynamic contributions to the generalised forces (equations (4-93), (4-94) and (4-100)) are not restricted to the two-dimensional strip theory aerodynamic approximation. As remarked in section 4.23 they can be used in general provided the appropriate meaning is given to the various aerodynamic coefficients.

4.5 Strip interference

An important factor affecting the dynamical behaviour of an aircraft, particularly when it is moving largely as a rigid body, is the aerodynamic interference between wing and tail. This is a particular manifestation of the fact that there is in general such interference between any two parts of the aircraft, *ie* in our representation, between any two strips. So far, in our analysis, we have largely ignored such interference. The various formulae that we have obtained can, as we have pointed out, be used with three-dimensional values for the aerodynamic coefficients, and so full account* can be taken of the interference effects. For example, the expression (4-21) for the overall forces on a strip referred to the strip-fixed axes could be extended to include terms

* Provided of course, that one has a useable aerodynamic theory for the whole configuration. Refs 8 and 9, for example, go a good way towards providing this for the subsonic case.

proportional to the perturbations of the velocities and control angle of the other strips - one or more of them - and the perturbations of their relative positions and orientation. Thus we can write

$$
\begin{aligned}
\begin{bmatrix} X_i^{(s)} \\ Y_i^{(s)} \\ Z_i^{(s)} \end{bmatrix} \approx \begin{bmatrix} X_{if}^{(s)} \\ 0 \\ Z_{if}^{(s)} \end{bmatrix} + \sum_j \Bigg\{ & \begin{bmatrix} \hat{X}^{(s)}_{i\dot{x}j} & \hat{X}^{(s)}_{i\dot{z}j} & \hat{X}^{(s)}_{i\dot{\theta}j} & \hat{X}^{(s)}_{i\delta j} \\ 0 & 0 & 0 & 0 \\ \hat{Z}^{(s)}_{i\dot{x}j} & \hat{Z}^{(s)}_{i\dot{z}j} & \hat{Z}^{(s)}_{i\dot{\theta}j} & \hat{Z}^{(s)}_{i\delta j} \end{bmatrix} \begin{bmatrix} u_j^{(s)} - u_{jf}^{(us)} \\ w_j^{(s)} - w_{jf}^{(us)} \\ q_j^{(s)} \\ \delta_j - \delta_{jf} \end{bmatrix} \\
& + \begin{bmatrix} \hat{X}^{(s)}_{ijx} & \hat{X}^{(s)}_{ixy} & \hat{X}^{(s)}_{ijz} & \hat{X}^{(s)}_{ij\phi} & \hat{X}^{(s)}_{ij\theta} & \hat{X}^{(s)}_{ij\psi} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hat{Z}^{(s)}_{ijx} & \hat{Z}^{(s)}_{ijy} & \hat{Z}^{(s)}_{ijz} & \hat{Z}^{(s)}_{ij\phi} & \hat{Z}^{(s)}_{ij\theta} & \hat{Z}^{(s)}_{ij\psi} \end{bmatrix} \begin{bmatrix} \Delta x_{ij} \\ \Delta y_{ij} \\ \Delta z_{ij} \\ \Delta \phi_{ij} \\ \Delta \theta_{ij} \\ \Delta \psi_{ij} \end{bmatrix} \Bigg\}
\end{aligned}
\qquad \text{...... (4-101)}
$$

and similar expressions for the moments and hinge moment on the ith strip. In these expressions Δx_{ij} etc are measures of the change in the relative position and orientation of the ith and jth strips. The coefficients $\hat{X}^{(s)}_{i\dot{x}j}$ etc will be functions of the position and orientation of the ith strip (ith strip-fixed axes) relative to the other strips (in the other strip-fixed axes) in the unperturbed state. The coefficients in the last term, $\hat{X}^{(s)}_{ijx}$ etc, can be taken as constants - the coefficients in the middle term, $\hat{X}^{(s)}_{i\dot{x}j}$ etc, will in general be differential operators - since any differential operator terms can be included in the other coefficients.

4.5.1 Using two-dimensional theory

Equation (4-101) indicates the possibility of making use of two-dimensional theory for interfering surfaces when one hasn't an adequate three-dimensional theory. It is, however, difficult to do this in general. A rather crude

approximate way of doing this is therefore suggested below. In certain circumstances it will become a good approximation, and in any case it should be in the right street.

If the y strip-fixed axes, of the ith and jth strips, were parallel, and the x strip-fixed axes were coplanar, in the datum state, and the datum flight path direction was parallel to that plane, then we could consider these strips as part of a two-dimensional configuration and obtain the aerodynamic interference accordingly. However, this is an exceptional situation. The way the aircraft has been divided into strips ensures that the last two conditions are satisfied when

$$y_{if} = y_{jf} \quad , \tag{4-102}$$

but to satisfy the other condition, we must have

$$\begin{cases} \text{either} \quad \phi_{if} = \phi_{jf} = 0 & \text{(4-103)} \\ \text{or} \quad \phi_{if} = \phi_{jf} \quad \text{and} \quad \theta_{if} = \theta_{jf} \quad . & \text{(4-104)} \end{cases}$$

It is therefore suggested, as a rough approximation, that one should, for any pair of strips which satisfy (4-102), obtain (two-dimensional) interference aerodynamic forces, assuming (4-103) was true, and then multiply these* by $\cos(\phi_i - \phi_j)$. Thus (4-101) is rewritten as (*cf* equation (3-3))

* The difference, between a force in the isolated and tandem configurations, is multiplied by $\cos(\phi_i - \phi_j)$ and then added to the force for the isolated configuration.

$$
\begin{bmatrix} X_i^{(s)} \\ Y_i^{(s)} \\ Z_i^{(s)} \end{bmatrix} \approx \left\{ \begin{bmatrix} X_{if}^{(2)} \\ 0 \\ Z_{if}^{(2)} \end{bmatrix} + \begin{bmatrix} \hat{X}_{i\dot{x}}^{(2)} & \hat{X}_{i\dot{z}}^{(2)} & \hat{X}_{i\dot{\theta}}^{(2)} & \hat{X}_{i\delta}^{(2)} \\ 0 & 0 & 0 & 0 \\ \hat{Z}_{i\dot{x}}^{(2)} & \hat{Z}_{i\dot{z}}^{(2)} & \hat{Z}_{i\dot{\theta}}^{(2)} & \hat{Z}_{i\delta}^{(2)} \end{bmatrix} \begin{bmatrix} u_i^{(s)} - u_{if}^{(us)} \\ w_i^{(s)} - w_{if}^{(us)} \\ q_i^{(s)} \\ \delta_i - \delta_{if} \end{bmatrix} \right\} \left(1 - \cos\left\{\phi_{if} - \phi_{jf}\right\}\right)
$$

$$
+ \left\{ \begin{bmatrix} X_{ijf}^{(2)} \\ 0 \\ Z_{ijf}^{(2)} \end{bmatrix} + \begin{bmatrix} \hat{X}_{ijx}^{(2)} & \hat{X}_{ijz}^{(2)} & \hat{X}_{ij\theta}^{(2)} \\ 0 & 0 & 0 \\ \hat{Z}_{ijx}^{(2)} & \hat{Z}_{ijz}^{(2)} & \hat{Z}_{ij\theta}^{(2)} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta z \\ \Delta\theta \end{bmatrix} \right.
$$

$$
\left. + \sum_{k=i,j} \begin{bmatrix} \hat{X}_{i\dot{x}k}^{(2)} & \hat{X}_{i\dot{z}k}^{(2)} & \hat{X}_{i\dot{\theta}k}^{(2)} & \hat{X}_{i\delta k}^{(2)} \\ 0 & 0 & 0 & 0 \\ \hat{Z}_{i\dot{x}k}^{(2)} & \hat{Z}_{i\dot{z}k}^{(2)} & \hat{Z}_{i\dot{\theta}k}^{(2)} & \hat{Z}_{i\delta k}^{(2)} \end{bmatrix} \begin{bmatrix} u_k^{(s)} - u_{kf}^{(us)} \\ w_k^{(s)} - w_{kf}^{(us)} \\ q_k^{(s)} \\ \delta_k - \delta_{kf} \end{bmatrix} \right\} \cos\left(\phi_{if} - \phi_{jf}\right)
$$

$$
+ \begin{bmatrix} X_{ijf}^{(2)} - X_{if}^{(2)} \\ 0 \\ Z_{ijf}^{(2)} - Z_{if}^{(2)} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} (F)_{i-j} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \sin\left(\phi_{if} - \phi_{jf}\right) \qquad (4\text{-}105)
$$

where the jth strip is one which satisfies equation (4-102) - we are assuming, as will almost certainly be the case, that there is only one such strip - the subscript (i - j) denotes the difference between the values of the modal function at the ith and jth strips, and

$$
\Delta x = \left\{\left(x_j^{(n)} - x_i^{(n)}\right)\cos\theta_i - \left(z_j^{(n)} - z_i^{(n)}\right)\sin\theta_i\right\} - \left\{\left(x_{jf} - x_{if}\right)\cos\theta_{if} - \left(z_{jf} - z_{if}\right)\sin\theta_{if}\right\} \qquad \ldots\ldots\ (4\text{-}106)
$$

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$$\Delta z = \left\{\left(z_j^{(n)} - z_i^{(n)}\right) \cos \theta_i + \left(x_j^{(n)} - x_i^{(n)}\right) \sin \theta_i\right\} - \left\{(z_{jf} - z_{if}) \cos \theta_{if} + (x_{jf} - x_{if}) \sin \theta_{if}\right\} \quad \ldots\ldots \text{(4-107)}$$

$$\Delta\theta = (\theta_j - \theta_i) - (\theta_{jf} - \theta_{if}) \quad . \qquad \text{(4-108)}$$

The use of equations (3-2), (3-3), (3-7) and (4-59) will enable one to write (4-105) entirely in terms of the generalised coordinates. The expressions for the moments and hinge moment at the ith strip will be similar to equation (4-105). The interference coefficients in the expressions, such as $X^{(2)}_{ijf}$, $\hat{X}^{(2)}_{i\dot{x}i}$, $\hat{Z}^{(2)}_{i\dot{x}j}$ etc, will be functions of x_f^{ij}, z_f^{ij}, θ_f^{ij}, u_{if}, w_{if}, δ_{if}, δ_{jf}, *inter alia* (*cf* Fig 2), where

$$x_f^{ij} = (x_{jf} - x_{if}) \cos \theta_{if} - (z_{jf} - z_{if}) \sin \theta_{if} \qquad \text{(4-109)}$$

$$z_f^{ij} = (z_{jf} - z_{if}) \cos \theta_{if} + (x_{jf} - x_{if}) \sin \theta_{if} \qquad \text{(4-110)}$$

$$\theta_f^{ij} = \theta_{jf} - \theta_{if} \quad . \qquad \text{(4-111)}$$

To avoid ambiguity, the coefficients with subscripts ijx, ijz or ijθ (*eg* $\hat{X}^{(2)}_{ijx}$) are taken to be constants. The other coefficients may be differential operators.

For some the significance, in some respects, of these descriptions, such as (4-105), for the forces and moments on a strip, may be illuminated if we consider the 'purely two-dimensional' case, when*

$$\left.\begin{aligned} \phi_{if} &= \psi_{if} = 0 \\ \phi_i &= \psi_i = 0 \\ v_f &= 0 \end{aligned}\right\} \qquad \text{(4-112)}$$

* From our specification of the datum motion (section 2) and the division into strips (section 3), ψ_{if} and v_f are always zero.

and similarly at the jth strip, and the perturbations are expressed as displacements relative to an inertial frame. No loss in generality is achieved by taking this frame to be the unperturbed strip-fixed axes (for each strip) and the body freedom displacements $x_1^{(c)}, \ldots, \phi, \ldots,$ to be zero. Then (*cf* equations (4-59), (3-2), (3-3), (3-7), (3-15) and footnote)

$$\begin{bmatrix} u_i^{(s)} - u_{if}^{(us)} \\ w_i^{(s)} - w_{if}^{(us)} \\ q_i^{(s)} \\ \delta_i - \delta_{if} \end{bmatrix} \approx \begin{bmatrix} D & 0 & -(w_f \cos\theta_{if} + u_f \sin\theta_{if}) & 0 \\ 0 & D & (u_f \cos\theta_{if} - w_f \sin\theta_{if}) & 0 \\ 0 & 0 & D & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i^{(us)} \\ z_i^{(us)} \\ \theta_i - \theta_{if} \\ \delta_i - \delta_{if} \end{bmatrix} . \qquad \ldots\ldots \text{(4-113)}$$

Also

$$\begin{bmatrix} \Delta x \\ \Delta z \\ \Delta\theta \end{bmatrix} = \begin{bmatrix} -1 & 0 & -z_f^{ij} & \cos\theta_f^{ij} & \sin\theta_f^{ij} & 0 \\ 0 & -1 & x_f^{ij} & -\sin\theta_f^{ij} & \cos\theta_f^{ij} & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i^{(us)} - x_{if}^{(us)} \\ z_i^{(us)} - z_{if}^{(us)} \\ \theta_i - \theta_{if} \\ x_j^{(us)} - x_{jf}^{(us)} \\ z_j^{(us)} - z_{jf}^{(us)} \\ \theta_j - \theta_{jf} \end{bmatrix} \qquad \ldots\ldots \text{(4-114)}$$

and so, for example, the forces on the ith strip, in the direction of the unperturbed strip-fixed axes, are:

$$\begin{bmatrix}X_i^{(us)}\\ Y_i^{(us)}\\ Z_i^{(us)}\end{bmatrix} \approx S^T_{\alpha_i}\begin{bmatrix}X_i^{(s)}\\ Y_i^{(s)}\\ Z_i^{(s)}\end{bmatrix} \approx \begin{bmatrix}X_{ijf}^{(2)}\\ 0\\ Z_{ijf}^{(2)}\end{bmatrix}$$

$$+ \begin{bmatrix} -\hat{X}^{(2)}_{ijx}+\hat{X}^{(2)}_{i\dot{x}i}D & -\hat{X}^{(2)}_{ijz}+\hat{X}^{(2)}_{i\dot{z}i}D & Z^{(2)}_{ijf} - \left(\hat{X}^{(2)}_{ij\theta} + z_f^{ij}\hat{X}^{(2)}_{ijx} - x_f^{ij}\hat{X}^{(2)}_{ijz}\right) - \hat{X}^{(2)}_{i\dot{x}i}(w_f\cos\theta_{if}+u_f\sin\theta_{if}) + \hat{X}^{(2)}_{i\dot{z}i}(u_f\cos\theta_{if} - w_f\sin\theta_{if}) + \hat{X}^{(2)}_{i\dot{\theta}i}D & \hat{X}_{i\delta i}\\ 0 & 0 & 0 & 0\\ -\hat{Z}^{(2)}_{ijx}+\hat{Z}^{(2)}_{i\dot{x}i}D & -\hat{Z}^{(2)}_{ijx}+\hat{Z}^{(2)}_{i\dot{z}i}D & -X^{(2)}_{ijf} - \left(\hat{Z}^{(2)}_{ij\theta} + z_f^{ij}\hat{Z}^{(2)}_{ijx} - x_f^{ij}\hat{Z}^{(2)}_{ijz}\right) - \hat{Z}^{(2)}_{i\dot{x}i}(w_f\cos\theta_{if}+u_f\sin\theta_{if}) + \hat{Z}^{(2)}_{i\dot{z}i}(u_f\cos\theta_{if} - w_f\sin\theta_{if}) + \hat{Z}^{(2)}_{i\dot{\theta}i}D & \hat{Z}_{i\delta i} \end{bmatrix} \begin{bmatrix} x_i^{(us)} - x_{if}^{(us)}\\ z_i^{(us)} - z_{if}^{(us)}\\ \theta_i - \theta_{if}\\ \delta_i - \delta_{if}\end{bmatrix}$$

$$+ \begin{bmatrix} \left(\hat{X}^{(2)}_{ijx}\cos\theta_f^{ij} - \hat{X}^{(2)}_{ijz}\sin\theta_f^{ij}\right) + \hat{X}^{(2)}_{i\dot{x}j}D & \left(\hat{X}^{(2)}_{ijz}\cos\theta_f^{ij} + \hat{X}^{(2)}_{ijx}\sin\theta_f^{ij}\right) + \hat{X}^{(2)}_{i\dot{z}j}D & \hat{X}^{(2)}_{ij\theta} - \hat{X}_{i\dot{x}j}(w_f\cos\theta_{jf}+u_f\sin\theta_{jf}) + \hat{X}^{(2)}_{i\dot{z}j}(u_f\cos\theta_{jf} - w_f\sin\theta_{jf}) + \hat{X}_{i\dot{\theta}j}D & \hat{X}^{(2)}_{i\delta j}\\ 0 & 0 & 0 & 0\\ \left(\hat{Z}^{(2)}_{ijx}\cos\theta_f^{ij} - \hat{Z}^{(2)}_{ijz}\sin\theta_f^{ij}\right) + \hat{Z}^{(2)}_{i\dot{x}j}D & \left(\hat{Z}^{(2)}_{ijz}\cos\theta_f^{ij} + \hat{Z}^{(2)}_{ijx}\sin\theta_f^{ij}\right) + \hat{Z}^{(2)}_{i\dot{z}j}D & \hat{Z}^{(2)}_{ij\theta} - \hat{Z}^{(2)}_{i\dot{x}j}(w_f\cos\theta_{jf}+u_f\sin\theta_{jf} + \hat{Z}^{(2)}_{i\dot{z}j}(u_f\cos\theta_{jf} - w_f\sin\theta_{jf}) + \hat{Z}^{(2)}_{i\dot{\theta}j}D & \hat{Z}^{(2)}_{i\delta j} \end{bmatrix} \begin{bmatrix} x_j^{(us)} - x_{jf}^{(us)}\\ z_j^{(us)} - z_{jf}^{(us)}\\ \theta_j - \theta_{jf}\\ \delta_j - \delta_{jf}\end{bmatrix}$$

...... (4-115)

This can be written as

$$\begin{bmatrix}X_i^{(us)}\\ Y_i^{(us)}\\ Z_i^{(us)}\end{bmatrix} \approx \begin{bmatrix}X_{ijf}^{(2)}\\ 0\\ Z_{ijf}^{(2)}\end{bmatrix} + \sum_{k=i,j} \begin{bmatrix} X^{(2)}_{ixk} & X^{(2)}_{izk} & X^{(2)}_{i\theta k} & X^{(2)}_{i\delta k}\\ 0 & 0 & 0 & 0\\ Z^{(2)}_{ixk} & Z^{(2)}_{izk} & Z^{(2)}_{i\theta k} & Z^{(2)}_{i\delta k}\end{bmatrix} \begin{bmatrix} x_k^{(us)} - x_{kf}^{(us)}\\ z_k^{(us)} - z_{kf}^{(us)}\\ \theta_k - \theta_{kf}\\ \delta_k - \delta_{kf}\end{bmatrix}$$

...... (4-116)

in the 'purely two-dimensional case', where, with $j \neq i$

$$X^{(2)}_{ixi} = -\hat{X}^{(2)}_{ijx} + \hat{X}^{(2)}_{i\dot{x}i}D \qquad (4\text{-}117)$$

$$X^{(2)}_{jxj} = \hat{X}^{(2)}_{ijx}\cos\theta_f^{ij} - \hat{X}^{(2)}_{ijz}\sin\theta_f^{ij} + \hat{X}^{(2)}_{i\dot{x}j}D \qquad (4\text{-}118)$$

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$$X^{(2)}_{izi} = -\hat{X}^{(2)}_{ijz} + \hat{X}^{(2)}_{i\dot{z}i}D \tag{4-119}$$

$$X^{(2)}_{izj} = \hat{X}^{(2)}_{ijz}\cos\theta^{ij}_f + \hat{X}^{(2)}_{ijx}\sin\theta^{ij}_f + \hat{X}^{(2)}_{i\dot{z}j}D \tag{4-120}$$

$$\begin{aligned} X^{(2)}_{i\theta i} = {} & Z^{(2)}_{ijf} - \left(\hat{X}^{(2)}_{ij\theta} + z^{ij}_f\hat{X}^{(2)}_{ijx} - x^{ij}_f\hat{X}^{(2)}_{ijz}\right) \\ & - (w_f\cos\theta_{if} + u_f\sin\theta_{if})\hat{X}^{(2)}_{i\dot{x}i} + (u_f\cos\theta_{if} - w_f\sin\theta_{if})\hat{X}_{i\dot{z}i} \\ & + \hat{X}^{(2)}_{i\dot{\theta}i}D \end{aligned} \tag{4-121}$$

$$\begin{aligned} X^{(2)}_{i\theta j} = {} & \hat{X}^{(2)}_{ij\theta} - (w_f\cos\theta_{jf} + u_f\sin\theta_{jf})\hat{X}^{(2)}_{i\dot{x}j} + (u_f\cos\theta_{jf} - w_f\sin\theta_{jf})\hat{X}^{(2)}_{i\dot{z}j} \\ & + \hat{X}^{(2)}_{i\dot{\theta}j}D \end{aligned} \tag{4-122}$$

$$X^{(2)}_{i\delta i} = \hat{X}^{(2)}_{i\delta i} \tag{4-123}$$

$$X^{(2)}_{i\delta j} = \hat{X}^{(2)}_{i\delta j} \tag{4-124}$$

and similarly for the Z coefficients with in particular

$$\begin{aligned} Z^{(2)}_{i\theta i} = {} & -X^{(2)}_{ijf} - \left(\hat{Z}^{(2)}_{ij\theta} + z^{ij}_f\hat{Z}^{(2)}_{ijx} - x^{ij}_f\hat{Z}^{(2)}_{ijz}\right) \\ & - (w_f\cos\theta_{if} + u_f\sin\theta_{if})\hat{Z}^{(2)}_{i\dot{x}i} + (u_f\cos\theta_{if} - w_f\sin\theta_{if})\hat{Z}^{(2)}_{i\dot{z}i} \\ & + \hat{Z}^{(2)}_{i\dot{\theta}i}D \quad . \end{aligned} \tag{4-125}$$

These equations enable one to derive the 'circumflexed' coefficients from the 'uncircumflexed' coefficients. If we introduce the notation*

$$\varepsilon(\ ) = \text{steady* part of } (\ ) \tag{4-126}$$

$$\partial(\ ) = (1-\varepsilon)(\qquad)D^{-1} \tag{4-127}$$

---

* Thus for $\quad F = F_0 + F_1D + F_2D^2$

$$\varepsilon(F) = F_0$$

$$\partial(F) = F_1 + F_2D \quad .$$

and write, $j \neq 1$

$$X^*_{i\theta i} = X^{(2)}_{i\theta i} + (w_f \cos\theta_{if} + u_f \sin\theta_{if})\partial\left(X^{(2)}_{ixi}\right) - (u_f \cos\theta_{if} - w_f \sin\theta_{if})\partial\left(X^{(2)}_{izi}\right)$$

$$- z_f^{ij}\varepsilon\left(X^{(2)}_{ixi}\right) + x_f^{ij}\varepsilon\left(X^{(2)}_{izi}\right) - Z^{(2)}_{ijf} \quad (4\text{-}128)$$

$$X^*_{i\theta j} = X^{(2)}_{i\theta j} + (w_f \cos\theta_{jf} + u_f \sin\theta_{jf})\partial\left(X^{(2)}_{ixj}\right) - (u_f \cos\theta_{jf} - w_f \sin\theta_{jf})\partial\left(X^{(2)}_{izj}\right)$$

...... (4-129)

$$Z^*_{i\theta i} = Z^{(2)}_{i\theta i} + (w_f \cos\theta_{if} + u_f \sin\theta_{if})\partial\left(Z^{(2)}_{ixi}\right) - (u_f \cos\theta_{if} - w_f \sin\theta_{if})\partial\left(Z^{(2)}_{izi}\right)$$

$$- z_f^{ij}\varepsilon\left(Z^{(2)}_{ixi}\right) + x_f^{ij}\varepsilon\left(Z^{(2)}_{izi}\right) + X^{(2)}_{ijf} \quad (4\text{-}130)$$

$$Z^*_{i\theta j} = Z^{(2)}_{i\theta j} + (w_f \cos\theta_{jf} + u_f \sin\theta_{jf})\partial\left(Z^{(2)}_{ixj}\right) - (u_f \cos\theta_{jf} - w_f \sin\theta_{jf})\partial\left(Z^{(2)}_{izj}\right) .$$

...... (4-131)

Then we have

$$\begin{bmatrix} \hat{X}^{(2)}_{ijx} & \hat{X}^{(2)}_{ijz} & \hat{X}^{(2)}_{ij\theta} \\ 0 & 0 & 0 \\ \hat{Z}^{(2)}_{ijx} & \hat{Z}^{(2)}_{ijz} & \hat{Z}^{(2)}_{ij\theta} \end{bmatrix} = \varepsilon \begin{bmatrix} -X^{(2)}_{ixi} & -X^{(2)}_{izi} & -X^*_{i\theta i} \\ 0 & 0 & 0 \\ -Z^{(2)}_{ixi} & -Z^{(2)}_{izi} & -Z^*_{i\theta i} \end{bmatrix}$$

$$= \varepsilon \begin{bmatrix} X^{(2)}_{ixj}\cos\theta_f^{ij} + X^{(2)}_{izj}\sin\theta_f^{ij} & X^{(2)}_{izj}\cos\theta_f^{ij} - X^{(2)}_{ixj}\sin\theta_f^{ij} & X^*_{i\theta j} \\ 0 & 0 & 0 \\ Z^{(2)}_{ixj}\cos\theta_f^{ij} + Z^{(2)}_{izj}\sin\theta_f^{ij} & Z^{(2)}_{izj}\cos\theta_f^{ij} - Z^{(2)}_{ixj}\sin\theta_f^{ij} & Z^*_{i\theta j} \end{bmatrix}$$

...... (4-132)

and

$$\begin{bmatrix} \hat{X}^{(2)}_{i\dot{x}k} & \hat{X}^{(2)}_{i\dot{z}k} & \hat{X}^{(2)}_{i\dot{\theta}k} \\ 0 & 0 & 0 \\ \hat{Z}^{(2)}_{i\dot{x}k} & \hat{Z}^{(2)}_{i\dot{z}k} & \hat{Z}^{(2)}_{i\dot{\theta}k} \end{bmatrix} = \partial \begin{bmatrix} X^{(2)}_{ixk} & X^{(2)}_{izk} & X^*_{i\theta k} \\ 0 & 0 & 0 \\ Z^{(2)}_{ixk} & Z^{(2)}_{izk} & Z^*_{i\theta k} \end{bmatrix} . \quad (4\text{-}133)$$

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These two equations, along with (4-123), (4-124) and their partners, give all the 'circumflexed' coefficients required in (4-105). Similar equations will give the coefficients appearing in the expressions for the moments and hinge moment.

For this 'purely two-dimensional' case the moment about the strip reference point referred to the unperturbed strip-fixed axes is

$$\begin{bmatrix} L_i^{(us)} \\ M_i^{(us)} \\ N_i^{(us)} \end{bmatrix} \approx S_{\alpha_i}^T \begin{bmatrix} 0 \\ M_i^{(s)} \\ 0 \end{bmatrix} \approx \begin{bmatrix} 0 \\ M_i^{(s)} \\ 0 \end{bmatrix} \qquad (4\text{-}134)$$

and so the moment about the origin of the unperturbed strip-fixed axes is

$$\begin{bmatrix} L_{ui}^{(us)} \\ M_{ui}^{(us)} \\ N_{ui}^{(us)} \end{bmatrix} = \begin{bmatrix} L_i^{(us)} \\ M_i^{(us)} \\ N_i^{(us)} \end{bmatrix} - A_{X_i^{(us)}} \begin{bmatrix} x_i^{(us)} - x_{if}^{(us)} \\ 0 \\ z_i^{(us)} - z_{if}^{(us)} \end{bmatrix}$$

$$\approx \begin{bmatrix} 0 \\ M_i^{(s)} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ X_{ijf}^{(2)}\left\{z_i^{(us)} - z_{if}^{(us)}\right\} - Z_{ijf}^{(2)}\left\{x_i^{(us)} - x_{if}^{(us)}\right\} \\ 0 \end{bmatrix} . \qquad (4\text{-}135)$$

We write this in a form similar to (4-116), *ie*

$$M_{ui}^{(us)} \approx M_{ijf}^{(2)} + \sum_{k=i,j} \begin{bmatrix} M_{ixk}^{(2)} & M_{izk}^{(2)} & M_{i\theta k}^{(2)} & M_{i\delta k}^{(2)} \end{bmatrix} \begin{bmatrix} x_k^{(us)} - x_{kf}^{(us)} \\ z_k^{(us)} - z_{kf}^{(us)} \\ \theta_k - \theta_{kf} \\ \delta_k - \delta_{kf} \end{bmatrix}$$

...... (4-136)

while writing $M_i^{(s)}$ in the form of (4-105) and using (4-113) and (4-114) gives, after some analysis, the relationships:

$$\begin{bmatrix}\hat{M}_{ijx}^{(2)} & \hat{M}_{ijz}^{(2)} & \hat{M}_{ij\theta}^{(2)}\end{bmatrix} = -\varepsilon\begin{bmatrix}M_{ixi}^{*} & M_{izi}^{*} & M_{i\theta i}^{*}\end{bmatrix}$$

$$= \varepsilon\begin{bmatrix}M_{ixj}^{(2)}\cos\theta_f^{ij} + M_{izj}^{(2)}\sin\theta_f^{ij} & M_{izj}^{(2)}\cos\theta_f^{ij} - M_{ixj}^{(2)}\sin\theta_f^{ij} & M_{i\theta j}^{*}\end{bmatrix}$$

...... (4-137)

$$\begin{bmatrix}\hat{M}_{i\dot{x}k}^{(2)} & \hat{M}_{i\dot{z}k}^{(2)} & \hat{M}_{i\dot{\theta}k}^{(2)}\end{bmatrix} = \partial\begin{bmatrix}M_{ixk}^{*} & M_{izk}^{*} & M_{i\theta k}^{*}\end{bmatrix} \qquad (4\text{-}138)$$

$$\hat{M}_{i\delta k}^{(2)} = M_{i\delta k}^{(2)} \qquad (4\text{-}139)$$

where

$$M_{ixk}^{*} = M_{ixi}^{(2)} + Z_{ijf}^{(2)} \qquad (k = i)$$

$$= M_{ixk}^{(2)} \qquad (k \neq i) \quad (4\text{-}140)$$

$$M_{izk}^{*} = M_{izi}^{(2)} - X_{ijf}^{(2)} \qquad (k = i)$$

$$= M_{izk}^{(2)} \qquad (k \neq i) \quad (4\text{-}141)$$

and

$$M_{i\theta k}^{*} = M_{i\theta i}^{(2)} + (w_f \cos\theta_{if} + u_f \sin\theta_{if})\partial\left(M_{ixi}^{(2)}\right)$$

$$- (u_f \cos\theta_{if} - w_f \sin\theta_{if})\partial\left(M_{izi}^{(2)}\right)$$

$$- z_f^{ij}\left\{Z_{ijf}^{(2)} + \varepsilon\left(M_{ixi}^{(2)}\right)\right\} - x_f^{ij}\left\{X_{ijf}^{(2)} - \varepsilon\left(M_{izi}^{(2)}\right)\right\} \qquad (k = i)$$

$$= M_{i\theta k}^{(2)} + (w_f \cos\theta_{kf} + u_f \sin\theta_{kf})\partial\left(M_{ixk}^{(2)}\right)$$

$$- (u_f \cos\theta_{kf} - w_f \sin\theta_{kf})\partial\left(M_{izk}^{(2)}\right) . \qquad (k \neq i) \quad (4\text{-}142)$$

The corresponding relationships for the hinge moment coefficients are easily seen to be

$$\begin{bmatrix}\hat{B}^{(2)}_{ijx} & \hat{B}^{(2)}_{ijz} & \hat{B}^{(2)}_{ij\theta}\end{bmatrix} = -\varepsilon\begin{bmatrix}B^{(2)}_{ixi} & B^{(2)}_{izi} & B^{*}_{i\theta i}\end{bmatrix}$$

$$= \varepsilon\begin{bmatrix}B^{(2)}_{ixj}\cos\theta^{ij}_{f} + B^{(2)}_{izj}\sin\theta^{ij}_{f} & B^{(2)}_{izj}\cos\theta^{ij}_{f} - B^{(2)}_{ixj}\sin\theta^{ij}_{f} & B^{*}_{i\theta j}\end{bmatrix}$$

$$\ldots\ldots \quad (4\text{-}143)$$

$$\begin{bmatrix}\hat{B}^{(2)}_{i\dot{x}k} & \hat{B}^{(2)}_{i\dot{z}k} & \hat{B}^{(2)}_{i\dot{\theta}k}\end{bmatrix} = \partial\begin{bmatrix}B^{(2)}_{ixk} & B^{(2)}_{izk} & B^{*}_{i\theta k}\end{bmatrix} \qquad (4\text{-}144)$$

$$\hat{B}^{(2)}_{i\delta k} = B^{(2)}_{i\delta k} \qquad (4\text{-}145)$$

where

$$\begin{aligned} B^{*}_{i\theta k} &= B^{(2)}_{i\theta i} + (w_f\cos\theta_{if} + u_f\sin\theta_{if})\partial\left(B^{(2)}_{ixi}\right) \\ &\quad - (u_f\cos\theta_{if} - w_f\sin\theta_{if})\partial\left(B^{(2)}_{izi}\right) \\ &\quad - z^{ij}_{f}\varepsilon\left(B^{(2)}_{ixi}\right) + x^{ij}_{f}\varepsilon\left(B^{(2)}_{izi}\right) \qquad (k = i) \\ &= B^{(2)}_{i\theta k} + (w_f\cos\theta_{kf} + u_f\sin\theta_{kf})\partial\left(B^{(2)}_{ixk}\right) \\ &\quad - (u_f\cos\theta_{kf} - w_f\sin\theta_{kf})\partial\left(B^{(2)}_{izk}\right) \qquad (k \neq i) \qquad (4\text{-}146)\end{aligned}$$

4.5.2 And in terms of the generalised coordinates

From equations (3-2) and (3-3), it is easily seen that

$$\begin{bmatrix}\Delta x \\ \Delta z \\ \Delta\theta\end{bmatrix} \approx \left\{\begin{bmatrix}\cos\theta_{if} & 0 & -\sin\theta_{if} \\ \sin\theta_{if} & 0 & \cos\theta_{if} \\ 0 & 0 & 0\end{bmatrix}(K)_{j-i} + \begin{bmatrix}0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0\end{bmatrix}(F)_j - \begin{bmatrix}0 & z^{ij}_{f} & 0 \\ 0 & -x^{ij}_{f} & 0 \\ 0 & 1 & 0\end{bmatrix}(F)_i\right\}\begin{bmatrix}q_1 \\ \vdots \\ q_n\end{bmatrix} .$$

$$\ldots\ldots \quad (4\text{-}147)$$

Equation (4-59) gives the other coordinates that we require in terms of the generalised coordinates. The coefficients in equation (4-105) are given by equations (4-123), (4-124), (4-132) and (4-133). Putting all these together we obtain the following expression for the overall forces on the ith strip:

$$
\begin{aligned}
\begin{bmatrix} X_i^{(s)} \\ Y_i^{(s)} \\ Z_i^{(s)} \end{bmatrix} \approx{}& \begin{bmatrix} X_{if}^{(2)} \\ 0 \\ Z_{if}^{(2)} \end{bmatrix} \Big(1 - \cos(\phi_{if} - \phi_{jf})\Big) + \begin{bmatrix} X_{ijf}^{(2)} \\ 0 \\ Z_{ijf}^{(2)} \end{bmatrix} \cos(\phi_{if} - \phi_{jf}) \\
&+ \Bigg[\Bigg\{ \partial \begin{bmatrix} X_{ix}^{(2)} & 0 & X_{iz}^{(2)} \\ 0 & 0 & 0 \\ Z_{ix}^{(2)} & 0 & Z_{iz}^{(2)} \end{bmatrix} S_{\phi_{if}} \begin{bmatrix} A_{u_f} S^T_{\phi_{if}} Q_{\phi_{if}} (F)_i + (K)_i D & ID & A_{u_f} - A_{x_{if}} D \end{bmatrix} \\
&\qquad + \partial \begin{bmatrix} 0 & X^*_{i\theta} & 0 \\ 0 & 0 & 0 \\ 0 & Z^*_{i\theta} & 0 \end{bmatrix} \begin{bmatrix} Q_{\phi_{if}} (F)_i D & 0 & S_{\phi_{if}} D \end{bmatrix} + \begin{bmatrix} X_{if}^{(2)} (f^T)_i & 0 & 0 \\ 0 & 0 & 0 \\ Z_{if}^{(2)} (f^T)_i & 0 & 0 \end{bmatrix} \Bigg\} \Big(1 - \cos(\phi_{if} - \phi_{jf})\Big) \\
&+ \Bigg\{ \sum_{k=i,j} \Bigg(\partial \begin{bmatrix} X_{ixk}^{(2)} & 0 & X_{izk}^{(2)} \\ 0 & 0 & 0 \\ Z_{ixk}^{(2)} & 0 & Z_{izk}^{(2)} \end{bmatrix} S_{\phi_{kf}} \begin{bmatrix} A_{u_f} S^T_{\phi_{kf}} Q_{\phi_{kf}} (F)_k + (K)_k D & ID & A_{u_f} - A_{x_{kf}} D \end{bmatrix} \\
&\qquad + \partial \begin{bmatrix} 0 & X^*_{i\theta k} & 0 \\ 0 & 0 & 0 \\ 0 & Z^*_{i\theta k} & 0 \end{bmatrix} \begin{bmatrix} Q_{\phi_{kf}} (F)_k D & 0 & S_{\phi_{kf}} D \end{bmatrix} + \begin{bmatrix} X_{i\delta k}^{(2)} (f^T)_k & 0 & 0 \\ 0 & 0 & 0 \\ Z_{i\delta k}^{(2)} (f^T)_k & 0 & 0 \end{bmatrix} \Bigg) - \varepsilon \begin{bmatrix} X_{ixi}^{(2)} & X_{izi}^{(2)} & X^*_{i\theta i} \\ 0 & 0 & 0 \\ Z_{ixi}^{(2)} & Z_{izi}^{(2)} & Z^*_{i\theta i} \end{bmatrix} \times \\
&\qquad \times \Bigg(\begin{bmatrix} \cos\theta_{if} & 0 & -\sin\theta_{if} \\ \sin\theta_{if} & 0 & \cos\theta_{if} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} (K)_{j-i} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} (F)_j & 0 & 0 \end{bmatrix} \\
&\qquad\qquad - \begin{bmatrix} 0 & z_f^{ij} & 0 \\ 0 & -x_f^{ij} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} (F)_i & 0 & 0 \end{bmatrix} \Bigg) \Bigg\} \cos(\phi_{if} - \phi_{jf}) \\
&+ \begin{bmatrix} X_{ijf}^{(2)} - X_{if}^{(2)} \\ 0 \\ Z_{ijf}^{(2)} - Z_{if}^{(2)} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} (F)_{i-j} & 0 & 0 \end{bmatrix} \sin(\phi_{if} - \phi_{jf}) \Bigg] \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \quad . \qquad (4\text{-}148)
\end{aligned}
$$

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The interference coefficients (*ie* all the ones with a triple subscript) in the above expression will be functions of x_f^{ij}, z_f^{ij}, θ_f^{ij}, u_{if}, w_{if}, δ_{if} and δ_{jf} (*cf* equations (4-109) to (4-111). Similar expressions can be obtained for the moments and hinge moment on a strip, to take some account of the aerodynamic interference between strips, using the relationships (4-137) to (4-139) and (4-143) to (4-145).

5 THE OTHER CONTRIBUTIONS TO THE GENERALISED FORCES

The contributions to the generalised forces* from the other forces* (gravitational, structural, etc) acting on the aircraft can be obtained, as is most convenient, either by obtaining expressions for the translational forces, moments and hinge moment on a strip and proceeding as we have done for the aerodynamic forces (*cf* section 4.4 and in particular, equations (4-82), (4-89) and (4-91)), or by substituting the expressions for the local forces and modal functions in the formula obtained in Ref 1 (section 6.1) and extended, for our form of deformation, in Appendix A of this paper (equations (A-12) and (A-13). With the latter approach, the modal functions required are (see equations (3-20) and (A-7)):

$$R = K - S^T_{\phi_{if}} A_{x_{sif}}^{(us)} Q_{\phi_{if}} F + S^T_{\phi_{if}} P^T_{\delta_{if}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{sie} - x_{hsi} \\ y_{sie} \\ z_{sie} \end{bmatrix} f^T \qquad (5\text{-}1)$$

at the ith strip ,

$$M_0 = S^T_{\phi_{if}} Q_{\phi_{if}} F \qquad (5\text{-}2)$$

$$N_0 = S^T_{\phi_{if}} \left\{ -A_{x_{sif}}^{(us)} Q_{\phi_{if}} F + 2P^T_{\delta_{if}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{sie} - x_{hsi} \\ y_{sie} \\ z_{sie} \end{bmatrix} f^T \right\} , \qquad (5\text{-}3)$$

---

* The word force, without the adjective effective, is to be understood as meaning applied force. The distinction is between applied forces and reversed effective forces (*cf* section 6).

$$M_1 = S^T_{\phi_{if}} P^T_{\delta_{if}} \begin{bmatrix} 0 \\ f^T \\ 0 \end{bmatrix} \tag{5-4}$$

$$N_1 = - S^T_{\phi_{if}} P^T_{\delta_{if}} A_{x_{sie}-x_{hsi}} \begin{bmatrix} 0 \\ f^T \\ 0 \end{bmatrix} \tag{5-5}$$

and

$$U = \cos \theta_{if} S^T_{\phi_{if}} A_{x^{(us)}_{sif}} \left(Q^T_{\phi_{if}}\right)^{-1} \quad . \tag{5-6}$$

With the former approach, that is proceeding as in section 4.4, if typical forces, moments and hinge moment on a strip are, referred to the strip-fixed axes,

$$\begin{bmatrix} \bar{X}^{(s)}_i \\ \bar{Y}^{(s)}_i \\ \bar{Z}^{(s)}_i \end{bmatrix} = \begin{bmatrix} \bar{X}^{(us)}_{if} \\ \bar{Y}^{(us)}_{if} \\ \bar{Z}^{(us)}_{if} \end{bmatrix} + \sum_{k=1}^{n+6} \begin{bmatrix} \bar{X}_{ik} \\ \bar{Y}_{ik} \\ \bar{Z}_{ik} \end{bmatrix} q_k \tag{5-7}$$

$$\begin{bmatrix} \bar{L}^{(s)}_i \\ \bar{M}^{(s)}_i \\ \bar{N}^{(s)}_i \end{bmatrix} = \begin{bmatrix} \bar{L}^{(us)}_{if} \\ \bar{M}^{(us)}_{if} \\ \bar{N}^{(us)}_{if} \end{bmatrix} + \sum_{k=1}^{n+6} \begin{bmatrix} \bar{L}_{ik} \\ \bar{M}_{ik} \\ \bar{N}_{ik} \end{bmatrix} q_k \tag{5-8}$$

and

$$\bar{B}_i = \bar{B}_{if} + \sum_{k=1}^{n+6} \bar{B}_{ik} q_k \tag{5-9}$$

then the contributions of these forces to the generalised forces are found to be

$$\bar{Q}^{(i)}_j \approx \bar{Q}^{(i)}_{jf} + \sum_{k=1}^{n+6} \bar{Q}^{(i)}_{jk} q_k \tag{5-10}$$

where

$$\begin{bmatrix} \bar{Q}^{(i)}_{1f} \\ \vdots \\ \bar{Q}^{(i)}_{n+6,f} \end{bmatrix} = \begin{bmatrix} K^T S^T_{\phi_{if}} \\ S^T_{\phi_{if}} \\ A_{x_{if}} S^T_{\phi_{if}} \end{bmatrix} \begin{bmatrix} \bar{X}^{(us)}_{if} \\ \bar{Y}^{(us)}_{if} \\ \bar{Z}^{(us)}_{if} \end{bmatrix} + \begin{bmatrix} F^T Q^T_{\phi_{if}} \\ 0 \\ S^T_{\phi_{if}} \end{bmatrix} \begin{bmatrix} \bar{L}^{(us)}_{if} \\ \bar{M}^{(us)}_{if} \\ \bar{N}^{(us)}_{if} \end{bmatrix} + \begin{bmatrix} f\bar{B}_{if} \\ 0 \\ 0 \end{bmatrix} \tag{5-11}$$

and

$$\bar{Q}^{(i)}_{jk} = \begin{bmatrix} K^T S^T_{\phi_{if}} \\ S^T_{\phi_{if}} \\ A_{x_{if}} S^T_{\phi_{if}} \end{bmatrix} \begin{bmatrix} \bar{X}_{i1} & \cdots & \bar{X}_{i,n+6} \\ \bar{Y}_{i1} & \cdots\cdots\cdots & \\ \bar{Z}_{i1} & \cdots\cdots\cdots & \end{bmatrix} + \begin{bmatrix} F^T Q^T_{\phi_{if}} \\ 0 \\ S^T_{\phi_{if}} \end{bmatrix} \begin{bmatrix} \bar{L}_{i1} & \cdots & \bar{L}_{i,n+6} \\ \bar{M}_{i1} & \cdots\cdots\cdots & \\ \bar{N}_{i1} & \cdots\cdots\cdots & \end{bmatrix} + \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \bar{B}_{i1} & \cdots & \bar{B}_{i,n+6} \end{bmatrix}$$

$$+ \begin{bmatrix} -K^T S^T_{\phi_{if}} A_{\bar{X}^{(us)}_{if}} Q_{\phi_{if}} F - \cos\theta_{if} F^T \mathcal{L}\left(Q^{-1}_{\phi_{if}} \begin{bmatrix} \bar{L}^{(us)}_{if} \\ \bar{M}^{(us)}_{if} \\ \bar{N}^{(us)}_{if} \end{bmatrix} \right) F & 0 & 0 \\ -S^T_{\phi_{if}} A_{\bar{X}^{(us)}_{if}} Q_{\phi_{if}} F & 0 & -S^T_{\phi_{if}} A_{\bar{X}^{(us)}_{if}} S_{\phi_{if}} \\ -A_{x_{if}} S^T_{\phi_{if}} A_{\bar{X}^{(us)}_{if}} Q_{\phi_{if}} F - S^T_{\phi_{if}} A_{\bar{L}^{(us)}_{if}} Q_{\phi_{if}} F & 0 & -\mathcal{L}\left(A_{x_{if}} S^T_{\phi_{if}} \begin{bmatrix} \bar{X}^{(us)}_{if} \\ \bar{Y}^{(us)}_{if} \\ \bar{Z}^{(us)}_{if} \end{bmatrix} + S^T_{\phi_{if}} \begin{bmatrix} \bar{L}^{(us)}_{if} \\ \bar{M}^{(us)}_{if} \\ \bar{N}^{(us)}_{if} \end{bmatrix} \right) \\ -S^T_{\phi_{if}} A_{\bar{X}^{(us)}_{if}} S_{\phi_{if}} K & & \end{bmatrix}$$

...... (5-12)

5.1 The gravitational contribution

The gravitational force on a particle is

$$\begin{bmatrix} e_g^{(c)} \\ f_g^{(c)} \\ g_g^{(c)} \end{bmatrix} = \delta m g \ell \Phi_f \tag{5-13}$$

referred to the constant-velocity axes where ℓ_{Φ_f} is the last column of S_{Φ_f} and the vector $\{\Phi_f \ \Theta_f \ \Psi_f\}$ is given by equation (2-2). Thus

$$\ell_{\Phi_f} = \begin{bmatrix} -\sin\Theta_f \\ \sin\Phi_f \cos\Theta_f \\ \cos\Phi_f \cos\Theta_f \end{bmatrix} = \begin{bmatrix} -\sin\Theta_f \\ 0 \\ \cos\Theta_f \end{bmatrix} . \tag{5-14}$$

Consequently the local gravitational force referred to the strip-fixed axes is

$$\begin{bmatrix} e_{gi}^{(s)} \\ f_{gi}^{(s)} \\ g_{gi}^{(s)} \end{bmatrix} = \delta mg S_{\phi_i} S\ell_{\Phi_f} \tag{5-15}$$

and the overall forces, moments, and hinge moment on a strip are, respectively, making use of equations (3-15) and (3-8), and defining m_i as the mass of the ith strip (*cf* equation (6-17)).

$$\begin{bmatrix} X_{gi}^{(s)} \\ Y_{gi}^{(s)} \\ Z_{gi}^{(s)} \end{bmatrix} = m_i g S_{\phi_i} S\ell_{\Phi_f}$$

$$\approx m_i g \left\{ S_{\phi_{if}} \ell_{\Phi_f} + \begin{bmatrix} S_{\phi_{if}} A_{\ell_{\Phi_f}} S^T_{\phi_{if}} Q_{\phi_{if}} F & 0 & S_{\phi_{if}} A_{\ell_{\Phi_f}} \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \right\} \tag{5-16}$$

$$\begin{bmatrix} L_{gi}^{(s)} \\ M_{gi}^{(s)} \\ N_{gi}^{(s)} \end{bmatrix} = g\left(\sum_{strip} \delta m A_{x_{si}^{(s)}}\right) S_{\phi_i} S^{\ell}\Phi_f$$

$$\approx m_i g \left\{ A_{a_{i0}} S_{\phi_{if}} {}^{\ell}\Phi_f + \left[A_{a_{i0}} S_{\phi_{if}} A_{\ell\Phi_f} S_{\phi_{if}}^T Q_{\phi_{if}} F - P_{\delta_{if}}^T A_{a_{i3}} P_{\delta_{if}} S_{\phi_{if}} {}^{\ell}\Phi_f f^T \quad 0 \quad A_{a_{i0}} S_{\phi_{if}} A_{\ell\Phi_f} \right] \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \right\}$$

...... (5-17)

$$B_{gi}^{(s)} = g[0 \quad 1 \quad 0]\left(\sum_{flap} \delta m A_{x_{si}^{(s)} - x_{hsi}}\right) S_{\phi_i} S^{\ell}\Phi_f$$

$$\approx m_i g \left\{ - a_{i3}^T P_{\delta_{if}} S_{\phi_{if}} {}^{\ell}\Phi_f + \left[- a_{i3}^T P_{\delta_{if}} S_{\phi_{if}} A_{\ell\Phi_f} S_{\phi_{if}}^T Q_{\phi_{if}} F - a_{i1}^T P_{\delta_{if}} S_{\phi_{if}} {}^{\ell}\Phi_f f^T \quad 0 \quad - a_{i3}^T P_{\delta_{if}} S_{\phi_{if}} A_{\ell\Phi_f} \right] \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \right\}$$

...... (5-18)

where, when the flap angle is zero, referred to the strip-fixed axes

the strip cg is at $\{x_{i0} \quad 0 \quad 0\}$

and the flap cg* is at $\{x_{hsi} + x_{i2} \quad 0 \quad 0\}$,

and where m_i, $\pi_i m_i$ are respectively the mass of the strip and of the flap part of the strip, and

* For an unmassbalanced flap one will expect x_{i2} to be negative.

$$a_{i1} = \pi_i x_{i2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{5-19}$$

$$a_{i3} = \pi_i x_{i2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{5-20}$$

$$a_{i0} = \begin{bmatrix} x_{i0} - \pi_i x_{i2} \\ 0 \\ 0 \end{bmatrix} + P^T_{\delta_{if}} a_{i1} \quad . \tag{5-21}$$

In this case, it is clearly much simpler to use the formulae of Appendix A (equations (A-12) and (A-13)) rather than carry out a similar analysis to that of section 4.4. Using (5-13) we then find that the gravitational contribution to the generalised force in the jth degree of freedom is

$$- G_j = - G_{jf} - \sum_{k=1}^{n+6} G_{jk} q_k \tag{5-22}$$

where
$$\begin{bmatrix} G_{1f} \\ \vdots \\ G_{n+6,f} \end{bmatrix} = - g \begin{bmatrix} \sum_i m_i \left(K^T + F^T Q^T_{\phi_{if}} A_{a_{i0}} S_{\phi_{if}} - f a^T_{i3} P_{\delta_{if}} S_{\phi_{if}} \right) \\ mI \\ \sum_i m_i A_{x_{if0}} \end{bmatrix} \ell_{\Phi_f} \tag{5-23}$$

with
$$\begin{bmatrix} x_{if0} \\ y_{if0} \\ z_{if0} \end{bmatrix} \equiv \begin{bmatrix} x_{if} \\ y_{if} \\ z_{if} \end{bmatrix} + S^T_{\phi_{if}} a_{i0} \tag{5-24}$$

and I a 3 × 3 unit matrix, and

$$[G_{jk}] = -g \sum_{i} m_i \begin{bmatrix} G^{(i)}_{qq} & 0 & G^{(i)}_{q\phi} \\ 0 & 0 & 0 \\ G^{(i)}_{\phi q} & 0 & G^{(i)}_{\phi\phi} \end{bmatrix} \tag{5-25}$$

and the non-zero submatrices in G_{jk} are

$$\begin{aligned} G^{(i)}_{qq} = {} & F^T \left\{ Q^T_{\phi_{if}} A_{a_{i0}} S_{\phi_{if}} A_{\ell\Phi_f} S^T_{\phi_{if}} Q_{\phi_{if}} - \cos\theta_{if}\, \mathcal{L}\left(Q^{-1}_{\phi_{if}} A_{a_{i0}} S_{\phi_{if}} \ell\Phi_f \right) \right\} F \\ & - F^T Q^T_{\phi_{if}} P^T_{\delta_{if}} A_{a_{i3}} P_{\delta_{if}} S_{\phi_{if}} \ell\Phi_f f^T \\ & - f a^T_{i3} P_{\delta_{if}} S_{\phi_{if}} A_{\ell\Phi_f} S^T_{\phi_{if}} Q_{\phi_{if}} F \\ & - f a^T_{i1} P_{\delta_{if}} S_{\phi_{if}} \ell\Phi_f f^T \end{aligned} \tag{5-26}$$

$$G^{(i)}_{q\phi} = \left\{ K^T + F^T Q^T_{\phi_{if}} A_{a_{i0}} S_{\phi_{if}} - f a^T_{i3} P_{\delta_{if}} S_{\phi_{if}} \right\} A_{\ell\Phi_f} \tag{5-27}$$

$$G^{(i)}_{\phi q} = G^{(i)^T}_{q\phi} \tag{5-28}$$

$$G^{(i)}_{\phi\phi} = A_{x_{if0}} A_{\ell\Phi_f} - \mathcal{L}\left(A_{x_{if0}} \ell\Phi_f \right) . \tag{5-29}$$

In obtaining these expressions use has been made of the general relationship, for any vectors $\{\phi \ldots\}$ and $\{x \ldots\}$,

$$Q_\phi^T A_x Q_\phi = \cos\theta \, A_{Q_\phi^{-1}\begin{bmatrix} x \\ y \\ z \end{bmatrix}}$$

$$= \cos\theta \left\{ \mathcal{L}_{Q_\phi^{-1}\begin{bmatrix} x \\ y \\ z \end{bmatrix}} - \mathcal{L}^T_{Q_\phi^{-1}\begin{bmatrix} x \\ y \\ z \end{bmatrix}} \right\} . \qquad (5\text{-}30)$$

5.2 The structural contribution

In this case it is simplest to use equations (4-91), (4-89) and (4-82) where the forces and moments in the latter equation are the structural forces, moments and hinge moment on a strip. These forces will be independent of the rigid body perturbation of the aircraft and so we can write them respectively as

$$\begin{bmatrix} X_{si}^{(s)} \\ Y_{si}^{(s)} \\ Z_{si}^{(s)} \end{bmatrix} = \begin{bmatrix} X_{sif}^{(s)} \\ Y_{sif}^{(s)} \\ Z_{sif}^{(s)} \end{bmatrix} + \sum_{k=1}^{n} \begin{bmatrix} X_{sik} \\ Y_{sik} \\ Z_{sik} \end{bmatrix} q_k \qquad (5\text{-}31)$$

$$\begin{bmatrix} L_{si}^{(s)} \\ M_{si}^{(s)} \\ N_{si}^{(s)} \end{bmatrix} = \begin{bmatrix} L_{sif}^{(s)} \\ M_{sif}^{(s)} \\ N_{sif}^{(s)} \end{bmatrix} + \sum_{k=1}^{n} \begin{bmatrix} L_{sik} \\ M_{sik} \\ N_{sik} \end{bmatrix} q_k \qquad (5\text{-}32)$$

and

$$B_{si}^{(s)} = B_{sif}^{(s)} + \sum_{k=1}^{n} B_{sik} q_k \quad . \qquad (5\text{-}33)$$

Now the structure cannot exert any overall force or moment on itself and so it is easily shown that

$$\sum_i S_{\phi_{if}}^T X_{sif}^{(s)} = 0 \qquad (5\text{-}34)$$

$$\sum_i S^T_{\phi_{if}} \left\{ \begin{bmatrix} X_{si1} & \cdots & X_{sin} \\ Y_{si1} & \cdots\cdots & \\ Z_{si1} & \cdots\cdots & \end{bmatrix} - A_{X^{(s)}_{sif}} Q_{\phi_{if}} F \right\} = 0 \qquad (5\text{-}35)$$

$$\sum_i \left\{ S^T_{\phi_{if}} L^{(s)}_{sif} + A_{x_{if}} S^T_{\phi_{if}} X^{(s)}_{sif} \right\} = 0 \qquad (5\text{-}36)$$

and

$$\sum_i \left(S^T_{\phi_{if}} \left\{ \begin{bmatrix} L_{si1} & \cdots & L_{sin} \\ M_{si1} & \cdots\cdots & \\ N_{si1} & \cdots\cdots & \end{bmatrix} - A_{L^{(s)}_{sif}} Q_{\phi_{if}} F - A_{X^{(s)}_{sif}} S_{\phi_{if}} K \right\} \right.$$

$$\left. + A_{x_{if}} S^T_{\phi_{if}} \left\{ \begin{bmatrix} X_{si1} & \cdots & X_{sin} \\ Y_{si1} & \cdots\cdots & \\ Z_{si1} & \cdots\cdots & \end{bmatrix} - A_{X^{(s)}_{sif}} Q_{\phi_{if}} F \right\} \right) = 0 \; . \qquad (5\text{-}37)$$

We then find that the structural contribution to the generalised forces is typically

$$- E_j = - E_{jf} - \sum_{k=1}^{n} E_{jk} q_k \qquad (5\text{-}38)$$

where

$$\begin{bmatrix} E_{1f} \\ \vdots \\ E_{n+6,f} \end{bmatrix} = - \sum_i \begin{bmatrix} K^T S^T_{\phi_{if}} X^{(s)}_{sif} + F^T Q^T_{\phi_{if}} L^{(s)}_{sif} + fB^{(s)}_{sif} \\ 0 \\ 0 \end{bmatrix} \qquad (5\text{-}39)$$

$$\begin{bmatrix} E_{jk} \end{bmatrix} = - \begin{bmatrix} E_{qq} \\ 0 \\ 0 \end{bmatrix} \qquad (5\text{-}40)$$

and the submatrix E_{qq} is

$$E_{qq} = \sum_i \left(K^T S^T_{\phi_{if}} \left\{ \begin{bmatrix} X_{si1} & \cdots & X_{sin} \\ Y_{si1} & \cdots\cdots & \\ Z_{si1} & \cdots\cdots & \end{bmatrix} - A_{X^{(s)}_{sif}} Q_{\phi_{if}} F \right\} \right.$$

$$+ F^T \left\{ Q^T_{\phi_{if}} \begin{bmatrix} L_{si1} & \cdots & L_{sin} \\ M_{si1} & \cdots\cdots & \\ N_{si1} & \cdots\cdots & \end{bmatrix} - \cos\theta_{if}\, \mathcal{L}\left(Q^{-1}_{\phi_{if}} L^{(s)}_{sif}\right) F \right\}$$

$$\left. + f \begin{bmatrix} B^{(s)}_{si1} & \cdots & B^{(s)}_{sin} \end{bmatrix} \right) . \qquad (5\text{-}41)$$

This all appears very simple, but of course quite a lot of work may be involved in getting the structural forces, $X^{(s)}_{si}$ etc, on a strip. If the strip can be considered as a slice of a beam then these forces and moments are the increments in the shearing forces, bending moments, etc, going from one side to the other of the strip.

5.3 The propulsive contribution

In Refs 1 and 2 a very simple model of the propulsive force was used. Improvement of that model is being considered, amongst other things, in another paper[6] currently being written. We will therefore in the present paper stick to the simple model though for convenience it is not quite the same model as that used previously[1,2]. Thus we assume that the propulsive force acting on a strip is such that it has constant components in the direction of the strip-fixed axes, and similarly for the propulsive moment about the strip reference point. That is, we assume

$$\begin{bmatrix} X^{(s)}_{pi} \\ Y^{(s)}_{pi} \\ Z^{(s)}_{pi} \end{bmatrix} = \begin{bmatrix} X^{(s)}_{pif} \\ Y^{(s)}_{pif} \\ Z^{(s)}_{pif} \end{bmatrix} \qquad (5\text{-}42)$$

$$\begin{bmatrix} L^{(s)}_{pi} \\ M^{(s)}_{pi} \\ N^{(s)}_{pi} \end{bmatrix} = \begin{bmatrix} L^{(s)}_{pif} \\ M^{(s)}_{pif} \\ N^{(s)}_{pif} \end{bmatrix} . \tag{5-43}$$

We also assume that there is no propulsive force or moment on the flap part of any strip.

The propulsive contribution to the generalised force in the jth degree of freedom is then easily seen to be

$$- P_j = - P_{jf} - \sum_{k=1}^{n+6} P_{jk} q_k \tag{5-44}$$

where (*cf* section 4.1)

$$\begin{bmatrix} P_{1f} \\ \vdots \\ P_{n+6,f} \end{bmatrix} = - \begin{bmatrix} \sum\limits_i \left\{ K^T S^T_{\phi\, if} \begin{bmatrix} X^{(s)}_{pif} \\ Y^{(s)}_{pif} \\ Z^{(s)}_{pif} \end{bmatrix} + F^T Q^T_{\phi\, if} \begin{bmatrix} L^{(s)}_{pif} \\ M^{(s)}_{pif} \\ N^{(s)}_{pif} \end{bmatrix} \right\} \\ \begin{bmatrix} X_{pf} \\ Y_{pf} \\ Z_{pf} \end{bmatrix} \\ \begin{bmatrix} L_{pf} \\ M_{pf} \\ N_{pf} \end{bmatrix} \end{bmatrix} \tag{5-45}$$

$$\begin{bmatrix} P_{jk} \end{bmatrix} = - \begin{bmatrix} P_{qq} & 0 & 0 \\ P_{xq} & 0 & P_{x\phi} \\ P_{\phi q} & 0 & P_{\phi\phi} \end{bmatrix} \tag{5-46}$$

the non-zero submatrices are

$$P_{qq} = \sum_i \left\{ - K^T S^T_{\phi_{if}} A_{X^{(s)}_{pif}} Q_{\phi_{if}} F - \cos\theta_{if} F^T \mathcal{L} \left(Q^{-1}_{\phi_{if}} \begin{bmatrix} L^{(s)}_{pif} \\ M^{(s)}_{pif} \\ N^{(s)}_{pif} \end{bmatrix} \right) \right\} \tag{5-47}$$

$$P_{xq} = \sum_i \left\{ - S^T_{\phi_{if}} A_{X^{(s)}_{pif}} Q_{\phi_{if}} F \right\} = \begin{bmatrix} X^{(c)}_{pq} \\ Y^{(c)}_{pq} \\ Z^{(c)}_{pq} \end{bmatrix} \tag{5-48}$$

$$P_{\phi q} = \sum_i \left\{ - S^T_{\phi_{if}} A_{X^{(s)}_{pif}} S_{\phi_{if}} K \right.$$

$$\left. - \left(A_{x_{if}} S^T_{\phi_{if}} A_{X^{(s)}_{pif}} + S^T_{\phi_{if}} A_{L^{(s)}_{pif}} \right) Q_{\phi_{if}} F \right\} = \begin{bmatrix} L^{(c)}_{pq} \\ M^{(c)}_{pq} \\ N^{(c)}_{pq} \end{bmatrix} \tag{5-49}$$

$$P_{x\phi} = \sum_i \left\{ - S^T_{\phi_{if}} A_{X^{(s)}_{pif}} S_{\phi_{if}} \right\} = - A_{X_{pf}} \tag{5-50}$$

$$P_{\phi\phi} = \sum_i \left\{ - \mathcal{L} \left(A_{x_{if}} S^T_{\phi_{if}} \begin{bmatrix} X^{(s)}_{pif} \\ Y^{(s)}_{pif} \\ Z^{(s)}_{pif} \end{bmatrix} + S^T_{\phi_{if}} \begin{bmatrix} L^{(s)}_{pif} \\ M^{(s)}_{pif} \\ N^{(s)}_{pif} \end{bmatrix} \right) \right\} = - \mathcal{L}_{L_{pf}} \tag{5-51}$$

and the overall propulsive forces and moments, about the origin of the constant-velocity axes, and referred to those axes are (*cf* section 4.3)

$$\begin{bmatrix} X_p^{(c)} \\ Y_p^{(c)} \\ Z_p^{(c)} \end{bmatrix} = \sum_i S^T_{\phi_{if}} \left\{ \begin{bmatrix} X^{(s)}_{pif} \\ Y^{(s)}_{pif} \\ Z^{(s)}_{pif} \end{bmatrix} - A_{X^{(s)}_{pif}} \begin{bmatrix} Q_{\phi_{if}} F & 0 & S_{\phi_{if}} \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \right\}$$

$$= \text{(say)} \begin{bmatrix} X_{pf} \\ Y_{pf} \\ Z_{pf} \end{bmatrix} + \begin{bmatrix} X^{(c)}_{pq} & X^{(c)}_{px} & X^{(c)}_{p\phi} \\ Y^{(c)}_{pq} & Y^{(c)}_{px} & Y^{(c)}_{p\phi} \\ Z^{(c)}_{pq} & Z^{(c)}_{px} & Z^{(c)}_{p\phi} \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \qquad (5\text{-}52)$$

$$\begin{bmatrix} L^{(c)}_{cp} \\ M^{(c)}_{cp} \\ N^{(c)}_{cp} \end{bmatrix} = \sum_i \left(\left\{ S^T_{\phi_{if}} \begin{bmatrix} L^{(s)}_{pif} \\ M^{(s)}_{pif} \\ N^{(s)}_{pif} \end{bmatrix} + A_{x_{if}} S^T_{\phi_{if}} \begin{bmatrix} X^{(s)}_{pif} \\ Y^{(s)}_{pif} \\ Z^{(s)}_{pif} \end{bmatrix} \right\} \right.$$

$$\left. + \begin{bmatrix} \begin{matrix} S^T_{\phi_{if}} \left\{ - A_{X^{(s)}_{pif}} S_{\phi_{if}} K - A_{L^{(s)}_{pif}} Q_{\phi_{if}} F \right\} \\ - A_{x_{if}} S^T_{\phi_{if}} A_{X^{(s)}_{pif}} Q_{\phi_{if}} F \end{matrix} & - S^T_{\phi_{if}} A_{X^{(s)}_{pif}} S_{\phi_{if}} & \begin{matrix} S^T_{\phi_{if}} \left\{ A_{X^{(s)}_{pif}} S_{\phi_{if}} A_{x_{if}} - A_{L^{(s)}_{pif}} S_{\phi_{if}} \right\} \\ - A_{x_{if}} S^T_{\phi_{if}} A_{X^{(s)}_{pif}} S_{\phi_{if}} \end{matrix} \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \right)$$

$$= \text{(say)} \begin{bmatrix} L_{pf} \\ M_{pf} \\ N_{pf} \end{bmatrix} + \begin{bmatrix} L^{(c)}_{pq} & L^{(c)}_{px} & L^{(c)}_{p\phi} \\ M^{(c)}_{pq} & M^{(c)}_{px} & M^{(c)}_{p\phi} \\ N^{(c)}_{pq} & N^{(c)}_{px} & N^{(c)}_{p\phi} \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \qquad (5\text{-}53)$$

The corresponding matrix of coefficients P_{jk} given in Ref 1 (Table 3) contained the matrix P_q of modal slopes at the aircraft reference point, *ie*

$$P_q = \left(\begin{bmatrix} 0 & 0 & \dfrac{\partial}{\partial y_f} \\ \dfrac{\partial}{\partial z_f} & 0 & 0 \\ 0 & \dfrac{\partial}{\partial x_f} & 0 \end{bmatrix} R \right)_{\text{reference point}} \quad . \qquad (5\text{-}54)$$

With our modal matrix R , equation (3-20), it is easily shown that

$$P_q = \left(S^T_{\phi_{if}} Q_{\phi_{if}} F\right)_0 \tag{5-55}$$

where the subscript 0 indicates that the expression is evaluated for the strip containing the reference point. If therefore, the only propulsive forces acting on the aircraft are acting on the main part of this strip, the submatrices in (5-46) have the particular values:

$$P_{qq} = -\left(K^T A_{X_{pf}} P_q - \cos\theta_{if} F^T \mathcal{L} \left(Q^{-1}_{\phi_{if}} S_{\phi_{if}} \left\{\begin{bmatrix} L_{pf} \\ M_{pf} \\ N_{pf} \end{bmatrix} - A_{x_{if}} \begin{bmatrix} X_{pf} \\ Y_{pf} \\ Z_{pf} \end{bmatrix}\right\}\right)\right)_0 \tag{5-56}$$

$$P_{xq} = -A_{X_{pf}} P_q \tag{5-57}$$

$$P_{\phi q} = -A_{X_{pf}} (K)_0 - \left\{A_{L_{pf}} + A_{X_{pf}} \left(A_{x_{if}}\right)_0\right\} P_q \tag{5-58}$$

$$P_{x\phi} = -A_{X_{pf}} \tag{5-59}$$

$$P_{\phi\phi} = -\mathcal{L}_{L_{pf}} \quad . \tag{5-60}$$

Thus, in this particular case, when the present assumption and that of Ref 1, as regards the propulsive forces, become identical, all these, except for P_{qq} , are, as expected, the same as the expressions* obtained in Ref 1, Table 3. The difference in the P_{qq} , as in the other contributions to the generalised forces, arises from the fact that the expressions for the deformations, here and in Ref 1, only agree to first order in the generalised coordinates (*cf* Appendix A).

---

* On the main part of the strip containing the reference point

$$R = \left(K - \left\{A_{x_f} - A_{x_{if}}\right\} P_q\right)_0 \quad .$$

6 THE GENERALISED EFFECTIVE FORCES

The velocity of the constant-velocity axes, resolved along the strip-fixed axes for the ith strip, is, using (3-15),

$$\begin{bmatrix} u_f^{(is)} \\ v_f^{(is)} \\ w_f^{(is)} \end{bmatrix} = S_{\phi_i} S \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix}$$

$$\approx S_{\phi_{if}} \left\{ \begin{bmatrix} u_f \\ v_f \\ w_f \end{bmatrix} + A_{u_f} S_{\phi_{if}}^T Q_{\phi_{if}} F \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + A_{u_f} \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \right\} . \qquad (6\text{-}1)$$

Consequently, the velocity of a particle relative to the constant velocity axes, and resolved along the strip-fixed axes, is, from (4-6),

$$\begin{bmatrix} u_{mi}^{(s)} - u_f^{(is)} \\ v_{mi}^{(s)} - v_f^{(is)} \\ w_{mi}^{(s)} - w_{mi}^{(is)} \end{bmatrix} \approx \left\{ S_{\phi_{if}} K - A_{x_{sif}^{(us)}} Q_{\phi_{if}} F + P_{\delta_{if}}^T \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{sie} - x_{hsi} \\ y_{sie} \\ z_{sie} \end{bmatrix} \right\} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix}$$

$$+ S_{\phi_{if}} \begin{bmatrix} \dot{x}_1^{(c)} \\ \dot{y}_1^{(c)} \\ \dot{z}_1^{(c)} \end{bmatrix} - \left(A_{x_{sif}^{(us)}} S_{\phi_{if}} + S_{\phi_{if}} A_{x_{if}} \right) \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

$$= \begin{bmatrix} S_{\phi_{if}} R & S_{\phi_{if}} & -\left(A_{x_{sif}^{(us)}} S_{\phi_{if}} + S_{\phi_{if}} A_{x_{if}} \right) \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_{n+6} \end{bmatrix} . \qquad (6\text{-}2)$$

The kinetic energy W, of the system, relative to the constant velocity axes, is given by

$$W = \tfrac{1}{2} \sum \delta m \begin{bmatrix} u_{mi}^{(s)} - u_f^{(is)} & v_{mi}^{(s)} - v_f^{(is)} & w_{mi}^{(s)} - w_f^{(is)} \end{bmatrix} \begin{bmatrix} u_{mi}^{(s)} - u_f^{(is)} \\ v_{mi}^{(s)} - v_f^{(is)} \\ w_{mi}^{(s)} - w_f^{(is)} \end{bmatrix} \qquad \ldots\ldots \text{(6-3)}$$

and so it is easily seen that the generalised effective forces are

$$\begin{bmatrix} \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{q}_1}\right) - \frac{\partial W}{\partial q_1} \\ \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{q}_2}\right) - \frac{\partial W}{\partial q_2} \\ \vdots \\ \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{q}_{n+6}}\right) - \frac{\partial W}{\partial q_{n+6}} \end{bmatrix} \approx \sum \delta m \begin{bmatrix} R^T S^T_{\phi_{if}} \\ S^T_{\phi_{if}} \\ S^T_{\phi_{if}} A_{x^{(us)}_{sif}} + A_{x_{if}} S^T_{\phi_{if}} \end{bmatrix} \times$$

$$\times \begin{bmatrix} S_{\phi_{if}} R & S_{\phi_{if}} & -\left(A_{x^{(us)}_{sif}} S_{\phi_{if}} + S_{\phi_{if}} A_{x_{if}}\right) \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \vdots \\ \ddot{q}_{n+6} \end{bmatrix}$$

$$= \text{(say)} \begin{bmatrix} A_{jk} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \vdots \\ \ddot{q}_{n+6} \end{bmatrix} . \qquad \text{(6-4)}$$

Now, we have from (5-21) and (5-24)

$$\sum \delta m A_{x_f} \equiv \sum \delta m \left(A_{x_{if}} + S^T_{\phi_{if}} A_{x^{(us)}_{sif}} S_{\phi_{if}}\right) = \sum_i m_i A_{x_{if0}} \qquad \text{(6-5)}$$

and we write*

$$- \sum \delta m A_{x_f}^2 \equiv - \sum \delta m \left(A_{x_{if}} + S_{\phi_{if}}^T A_{x_{sif}^{(us)}} S_{\phi_{if}} \right)^2 = I_n \quad . \qquad (6\text{-}6)$$

Thus, the inertia matrix is the symmetric matrix

$$\left[A_{jk} \right] = \begin{bmatrix} \sum \delta m R^T R & \sum \delta m R^T & - \sum \delta m R^T A_{x_f} \\ \sum \delta m R & mI & - \sum \delta m A_{x_f} \\ \sum \delta m A_{x_f} R & \sum \delta m A_{x_f} & I_n \end{bmatrix} \qquad (6\text{-}7)$$

where R is given by equation (3-20) and $\{x_f \; y_f \; z_f\}$ by equation (3-21). We can write the various submatrices in (6-7) in terms of the inertia properties of the strip though when transcribed at length the result is rather messy. To do so we write

$$\sum_{\text{strip}} \delta m A_{x_{sif}^{(us)}}^2 = - I_{i0} \equiv - \begin{bmatrix} I_x^{(i0)} & - I_{xy}^{(i0)} & - I_{xz}^{(i0)} \\ - I_{xy}^{(i0)} & I_y^{(i0)} & - I_{yz}^{(i0)} \\ - I_{xz}^{(i0)} & - I_{yz}^{(i0)} & I_z^{(i0)} \end{bmatrix} \qquad (6\text{-}8)$$

$$\sum_{\text{strip flap}} \delta m A_{(x_{sie} - s_{hsi})}^2 = - I_{i1} \equiv - \begin{bmatrix} I_x^{(i1)} & - I_{xy}^{(i1)} & - I_{xz}^{(i1)} \\ - I_{xy}^{(i1)} & I_y^{(i1)} & - I_{yz}^{(i1)} \\ - I_{xz}^{(i1)} & - I_{yz}^{(i1)} & I_z^{(i1)} \end{bmatrix} \qquad (6\text{-}9)$$

and make use of the definitions of section 5.1, equations (5-21) and (5-19), which give

---

* If the constant-velocity axes coincide with the principal axes of inertia during the datum motion then I_n will be diagonal $(= \text{diag}\{I_x \;\; I_y \;\; I_z\})$ where I_x, I_y, I_z are the principal moments of inertia of the aircraft in its datum state.

$$\sum_{\text{strip}} \delta m A_{x_{sif}}^{(us)} = m_i A_{a_{i0}} \tag{6-10}$$

$$\sum_{\text{strip flap}} \delta m A_{(x_{sie} - x_{hsi})} = m_i A_{a_{i1}} \quad . \tag{6-11}$$

In passing, we note that this means that

$$\sum_{\text{strip}} \delta m A^2_{x_{sie}} = - I_{i0} - m_i \left(A_{x_{hsi}} \left\{ P^T_{\delta_{if}} A_{a_{i1}} P_{\delta_{if}} - A_{a_{i1}} \right\} + \left\{ P^T_{\delta_{if}} A_{a_{i1}} P_{\delta_{if}} - A_{a_{i1}} \right\} A_{x_{hsi}} \right) + \left(P^T_{\delta_{if}} I_{i1} P_{\delta_{if}} - I_{i1} \right) \tag{6-12}$$

and so, as with a_{i0}, I_{i0} is a function of δ_{if}

$$I_{i0} = (I_{i0})_{\delta_{if}=0} - m_i \left(A_{x_{hsi}} \left\{ P^T_{\delta_{if}} A_{a_{i1}} P_{\delta_{if}} - A_{a_{i1}} \right\} + \left\{ P^T_{\delta_{if}} A_{a_{i1}} P_{\delta_{if}} - A_{a_{i1}} \right\} A_{x_{hsi}} \right) + \left(P^T_{\delta_{if}} I_{i1} P_{\delta_{if}} - I_{i1} \right) \quad . \tag{6-13}$$

With the definitions of equation (6-8) to (6-11) we find that the terms involving the deformation modes in (6-7) are

$$\sum \delta m R = \sum_i m_i \left\{ K - S^T_{\phi_{if}} A_{a_{i0}} Q_{\phi_{if}} F - S^T_{\phi_{if}} P^T_{\delta_{if}} A_{a_{i1}} \begin{bmatrix} 0 \\ f^T \\ 0 \end{bmatrix} \right\} \tag{6-14}$$

$$\sum \delta m A_{x_f} R = \sum_i \left(m_i \left\{ \left(A_{x_{if}} + S^T_{\phi_{if}} A_{a_{i0}} S_{\phi_{if}} \right) K - A_{x_{if}} S^T_{\phi_{if}} A_{a_{i0}} Q_{\phi_{if}} F \right. \right.$$

$$\left. - \left(A_{x_{if}} S^T_{\phi_{if}} + S^T_{\phi_{if}} A_{x_{hsi}} \right) P^T_{\delta_{if}} A_{a_{i1}} \begin{bmatrix} 0 \\ f^T \\ 0 \end{bmatrix} \right\}$$

$$\left. + S^T_{\phi_{if}} \left\{ I_{i0} Q_{\phi_{if}} F + P^T_{\delta_{if}} I_{i1} \begin{bmatrix} 0 \\ f^T \\ 0 \end{bmatrix} \right\} \right) \qquad (6\text{-}15)$$

$$\sum \delta m R^T R = \sum_i \left(m_i \left\{ K^T K - K^T S^T_{\phi_{if}} A_{a_{i0}} Q_{\phi_{if}} F \right. \right.$$

$$+ F^T Q^T_{\phi_{if}} A_{a_{i0}} S_{\phi_{if}} K - \left(K^T S^T_{\phi_{if}} + F^T Q^T_{\phi_{if}} A_{x_{hsi}} \right) P^T_{\delta_{if}} A_{a_{i1}} \begin{bmatrix} 0 \\ f^T \\ 0 \end{bmatrix}$$

$$\left. + \begin{bmatrix} 0 & f & 0 \end{bmatrix} A_{a_{i1}} P_{\delta_{if}} \left(S_{\phi_{if}} K + A_{x_{hsi}} Q_{\phi_{if}} F \right) \right\}$$

$$+ \left\{ F^T Q^T_{\phi_{if}} I_{i0} Q_{\phi_{if}} F + F^T Q^T_{\phi_{if}} P^T_{\delta_{if}} I_{i1} \begin{bmatrix} 0 \\ f^T \\ 0 \end{bmatrix} \right.$$

$$+ \begin{bmatrix} 0 & f & 0 \end{bmatrix} I_{i1} P_{\delta_{if}} Q_{\phi_{if}} F$$

$$\left. \left. + \begin{bmatrix} 0 & f & 0 \end{bmatrix} I_{i1} \begin{bmatrix} 0 \\ f^T \\ 0 \end{bmatrix} \right\} \right) \qquad (6\text{-}16)$$

The other terms are given by (6-5),

$$m = \sum_i m_i \tag{6-17}$$

and

$$I_n = \sum_i \left(S_{\phi_{if}}^T I_{i0} S_{\phi_{if}} - m_i \left\{ A_{x_{if}}^2 + A_{x_{if}} S_{\phi_{if}}^T A_{a_{i0}} S_{\phi_{if}} + S_{\phi_{if}}^T A_{a_{i0}} S_{\phi_{if}} A_{x_{if}} \right\} \right) . \tag{6-18}$$

7 THE EQUATIONS OF MOTION

Lagrange's equations merely equate the generalised forces and the generalised effective forces*. Stating this equality for our (n + 6) degrees of freedom under the assumption of small perturbations gives two sets of (n + 6) simultaneous equations. With the datum motion that we have taken - briefly constant linear velocity, zero angular velocity in a uniform atmosphere (*cf* section 2) - the first set state that the values of the generalised forces in the datum motion are all zero, *ie*

$$\bar{Q}_{rf} = 0 \qquad r = 1 \rightarrow (n+6) \tag{7-1}$$

or in terms of the separate contributions (aerodynamic, gravitational, etc)

$$- Q_{rf} + G_{rf} + P_{rf} + E_{rf} = 0 \qquad r = 1 \rightarrow n+6 . \tag{7-2}$$

Expressions for the individual elements in these equations have been obtained above - equations (4-93), (5-23), (5-45) and (5-39).

The second set of equations express the continued satisfaction of D'Alembert's principle during small perturbations of the datum motion. They are written as the matrix equation

* Often they are expressed on the equality of (on the left hand side) the generalised effective forces minus the conservative generalised forces, and (on the right hand side) the non-conservative generalised forces. This is the form when the equations are written in terms of the Lagrangian function.

$$\left\{[A_{rs}]D^2 + [G_{rs}] + [P_{rs}] + [E_{rs}] - [Q_{rs}]\right\}\begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} = 0 \qquad (7\text{-}3)$$

where the constituent matrices are:

(i) the inertia matrix $[A_{rs}]$, equation (6-7), where the various submatrices are given by equations (6-16), (6-14), (6-15), (6-17) and (6-18);

(ii) the gravitational matrix $[G_{rs}]$, equation (5-25), where the non-zero submatrices are given by equations (5-26) to (5-28);

(iii) the propulsive matrix $[P_{rs}]$, equation (5-45), where the various submatrices are given by equations (5-47) to (5-51);

(iv) the structural matrix $[E_{rs}]$, equation (5-40), which has one non-zero submatrix given by equation (5-41);

(v) the aerodynamic matrix $-[Q_{rs}]$, equations (4-94) and (4-100), see also sections 4.5, 4.5.1 and 4.5.2.

Equation (7-3) is almost the same* as the equation given in Table 3 of Ref 1 except that it has been written in terms of sectional properties and in terms of modal functions appropriate to a sectional description of the configuration. In particular a minor restriction is imposed on the form of deformation in that what might be called 'chordwise deformation' is forbidden apart from that due to a flap rotation; but more noticeable are the additional terms in the qq submatrices resulting from different ways used to express the deformation (*cf* Appendix A). Thus in Ref 1 the submatrix G_{qq}, for example, is null, whereas in the present development it is given by equation (5-26). There are consequently some corresponding differences in the transformation to other forms of the equations of motion. These differences have been demonstrated in detail in Appendix C.

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* When the choice of constant-velocity axes is the same. In Ref 1 they have been chosen to coincide with the principal axes of inertia during the datum motion. This is equivalent to putting $\sum m_i A_{x_{if0}} = 0$, and $I_n = \mathrm{diag}\{I_x \; I_y \; I_z\}$, in the present development.

8 CONCLUDING REMARKS

This paper has been written as a companion document to Ref 1. In each case the equations of motion of an aircraft, for small perturbations from flight with constant linear and zero angular velocity, have been developed in detail. The constrast has been that, whereas in Ref 1 we took an overall view of the aircraft, we have here taken a fore and aft strip of the aircraft as our basic unit, considered the forces, etc on the strip, and built up from that. As a consequence, the representation of the aircraft deformation in the present paper cannot be made to correspond exactly with that of the earlier paper. The deformational representation is basically more complicated; and other complications, such as local axes for each strip, are also introduced by strip approach. Of course a lot of these complications will disappear in the simplest cases (*cf* Ref 5) but even so, one would not recommend the use of the present method unless, as may quite possibly be the case, it means that adequate basic data can be obtained much more simply, as for example if two-dimensional aerodynamic theory can be used.

Appendix A

A GENERALISATION OF THE GENERALISED FORCE EXPRESSION OBTAINED IN REF 1

In Ref 1 it was assumed (equation (1)) that the deformation of the aircraft was given precisely by a first order expression, viz:

$$\begin{bmatrix} x_n^{(n)} \\ y_n^{(n)} \\ z_n^{(n)} \end{bmatrix} = \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} + R \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} . \tag{A-1}$$

This assumption simplified some of the subsequent development, while making no significant restriction to the freedom to represent, to a good approximation, any deformation of the aircraft. However, in certain cases, such as a deformation which involves rotation of part of the aircraft as a rigid body, it is not possible to represent the deformation exactly by (A-1) and still keep the number of degrees of freedom, n , finite. The deformations assumed in this paper are indeed of this type, and with the chosen degrees of freedom, their representation is (*cf* section 3)

$$\begin{bmatrix} x_{ni}^{(n)} \\ y_{ni}^{(n)} \\ z_{ni}^{(n)} \end{bmatrix} = \begin{bmatrix} x_{if} \\ y_{if} \\ z_{if} \end{bmatrix} + K \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + S_{\phi_i}^T \begin{bmatrix} x_{si}^{(s)} \\ y_{si}^{(s)} \\ z_{si}^{(s)} \end{bmatrix} . \tag{A-2}$$

If we make use of the expansions

$$S_{\phi_i}^T = S_{\phi_{if}}^T \left\{ I + A_{Q_{\phi_{if}}\begin{bmatrix} \phi_i-\phi_{if} \\ \theta_i-\theta_{if} \\ \psi_i-\psi_{if} \end{bmatrix}} + \frac{1}{2}\left[A^2_{Q_{\phi_{if}}\begin{bmatrix} \phi_i-\phi_{if} \\ \theta_i-\theta_{if} \\ \psi_i-\psi_{if} \end{bmatrix}} - \cos\theta_{if}\, A_{\left(Q_{\phi_{if}}^T\right)^{-1} J_{\begin{bmatrix} \phi_i-\phi_{if} \\ \theta_i-\theta_{if} \\ \psi_i-\psi_{if} \end{bmatrix}} \begin{bmatrix} \phi_i-\phi_{if} \\ \theta_i-\theta_{if} \\ \psi_i-\psi_{if} \end{bmatrix}} \right] + \ldots \right\}$$

...... (A-3)

$$P^T_{\delta_i} = P^T_{\delta_{if}} \left\{ I + (\delta_i - \delta_{if}) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + \tfrac{1}{2}(\delta_i - \delta_{if})^2 \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \dots \right\} \qquad \text{...... (A-4)}$$

along with the relationships

$$\begin{bmatrix} x^{(s)}_{si} \\ y^{(s)}_{si} \\ z^{(s)}_{si} \end{bmatrix} = \begin{bmatrix} x_{hsi} \\ 0 \\ 0 \end{bmatrix} + P^T_{\delta_i} \begin{bmatrix} x_{sie} - x_{hsi} \\ y_{sie} \\ z_{sie} \end{bmatrix} \qquad \text{(A-4a)}$$

on the flap part of the strip

$$\begin{bmatrix} \phi_i \\ \theta_i \\ \psi_i \end{bmatrix} = \begin{bmatrix} \phi_{if} \\ \theta_{if} \\ \psi_{if} \end{bmatrix} + F \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \qquad \text{(A-5)}$$

and

$$\delta_i = \delta_{if} + f^T \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \qquad \text{(A-6)}$$

then the second order approximation to the deformation is seen to be (using (3-6))

$$
\begin{aligned}
\begin{bmatrix} x_{ni}^{(n)} \\ y_{ni}^{(n)} \\ z_{ni}^{(n)} \end{bmatrix} \approx{}& \begin{bmatrix} x_{if} \\ y_{if} \\ z_{if} \end{bmatrix} + S_{\phi_{if}}^{T} \begin{bmatrix} x_{sif}^{(us)} \\ y_{sif}^{(us)} \\ z_{sif}^{(us)} \end{bmatrix} \\
&+ \left\{ K - S_{\phi_{if}}^{T} A_{x_{sif}^{(us)}} Q_{\phi_{if}} F + S_{\phi_{if}}^{T} P_{\delta_{if}}^{T} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{sie} - x_{hsi} \\ y_{sie} \\ z_{sie} \end{bmatrix} f^{T} \right\} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \\
&+ \frac{1}{2} \Bigg[A_{S_{\phi_{if}}^{T} Q_{\phi_{if}} F \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}} S_{\phi_{if}}^{T} \times \\
&\qquad \times \left\{ - A_{x_{sif}^{(us)}} Q_{\phi_{if}} F + 2 P_{\delta_{if}}^{T} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{sie} - x_{hsi} \\ y_{sie} \\ z_{sie} \end{bmatrix} f^{T} \right\} \\
&\qquad - A_{\left(S_{\phi_{if}}^{T} P_{\delta_{if}}^{T} \begin{bmatrix} 0 \\ f^{T} \\ 0 \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \right)} S_{\phi_{if}}^{T} P_{\delta_{if}}^{T} A_{(x_{sie} - x_{hsi})} \begin{bmatrix} 0 \\ f^{T} \\ 0 \end{bmatrix} \\
&\qquad + \cos \theta_{if} \, S_{\phi_{if}}^{T} A_{x_{sif}^{(us)}} \left(Q_{\phi_{if}}^{T} \right)^{-1} J_{F \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}} F \Bigg] \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} .
\end{aligned}
\qquad \text{...... (A-7)}
$$

This equation is a refinement of (3-18). It will be seen that the additional (*cf* with Ref 1) second order term has the form

$$\begin{bmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{z}\end{bmatrix} \text{(say)} = \left(A_{M_0}\begin{bmatrix}q_1\\ \vdots\\ q_n\end{bmatrix}^{N_0} + A_{M_1}\begin{bmatrix}q_1\\ \vdots\\ q_n\end{bmatrix}^{N_1} + UJ_F\begin{bmatrix}q_1\\ \vdots\\ q_n\end{bmatrix}^{F}\right)\begin{bmatrix}q_1\\ \vdots\\ q_n\end{bmatrix} \qquad \text{(A-8)}$$

where M_0, N_0, M_1, N_1 and U are functions of location.

Now,

$$\begin{bmatrix}\frac{\partial}{\partial q_1}[\tilde{x}\ \ \tilde{y}\ \ \tilde{z}]\\ \frac{\partial}{\partial q_2}[\tilde{x}\ \ \tilde{y}\ \ \tilde{z}]\\ \cdots\cdots\cdots\cdots\\ \frac{\partial}{\partial q_n}[\tilde{x}\ \ \tilde{y}\ \ \tilde{z}]\end{bmatrix}\begin{bmatrix}\bar{e}\\ \bar{f}\\ \bar{g}\end{bmatrix} = \left\{\sum_{s=0}^{1}\left(M_s^T A_{N_s}\begin{bmatrix}q_1\\ \vdots\\ q_n\end{bmatrix} - N_s^T A_{M_s}\begin{bmatrix}q_1\\ \vdots\\ q_n\end{bmatrix}\right) + F^T\left(J_F^T\begin{bmatrix}q_1\\ \vdots\\ q_n\end{bmatrix} - K_F^T\begin{bmatrix}q_1\\ \vdots\\ q_n\end{bmatrix}\right)U^T\right\}\begin{bmatrix}\bar{e}\\ \bar{f}\\ \bar{g}\end{bmatrix}$$

$$= \left\{\sum_{s=0}^{1}\left(N_s^T A_{\bar{e}} M_s - M_s^T A_{\bar{e}} N_s\right) + F^T\left(\mathcal{L}_{U^T\begin{bmatrix}\bar{e}\\ \bar{f}\\ \bar{g}\end{bmatrix}} + \mathcal{L}^T_{U^T\begin{bmatrix}\bar{e}\\ \bar{f}\\ \bar{g}\end{bmatrix}}\right)F\right\}\begin{bmatrix}q_1\\ \vdots\\ q_n\end{bmatrix} \qquad \text{(A-9)}$$

and so we can see that the additional terms (A-8) in the expression for the deformation produce additional terms only in the top left hand submatrix of the matrix of generalised force coefficients and this additional submatrix is

$$\sum\left\{\sum_{s=0}^{1}\left(N_s^T A_{\bar{e}_f} M_s - M_s^T A_{\bar{e}_f} N_s\right) + F^T\left(\mathcal{L}_{U^T\begin{bmatrix}\bar{e}_f\\ \bar{f}_f\\ \bar{g}_f\end{bmatrix}} + \mathcal{L}^T_{U^T\begin{bmatrix}\bar{e}_f\\ \bar{f}_f\\ \bar{g}_f\end{bmatrix}}\right)F\right\} . \qquad \text{(A-10)}$$

Thus, from equation (73) of Ref 1 the matrix of generalised coefficients resulting from a typical local force vector

$$\begin{bmatrix} \bar{e}^{(c)} \\ \bar{f}^{(c)} \\ \bar{g}^{(c)} \end{bmatrix} = \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} + \begin{bmatrix} \bar{e}_1 & \cdots & \bar{e}_{n+6} \\ \bar{f}_1 & \cdots & \cdots \\ \bar{g}_1 & \cdots & \cdots \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \tag{A-11}$$

is

$$[\bar{Q}_{ij}] = \begin{bmatrix} \sum R^T \begin{bmatrix} \bar{e}_1 & \cdots & \bar{e}_{n+6} \\ \bar{f}_1 & \cdots & \cdots \\ \bar{g}_1 & \cdots & \cdots \end{bmatrix} \\ \sum \begin{bmatrix} \bar{e}_1 & \cdots & \bar{e}_{n+6} \\ \bar{f}_1 & \cdots & \cdots \\ \bar{g}_1 & \cdots & \cdots \end{bmatrix} \\ \sum A_{x_f} \begin{bmatrix} \bar{e}_1 & \cdots & \bar{e}_{n+6} \\ \bar{f}_1 & \cdots & \cdots \\ \bar{g}_1 & \cdots & \cdots \end{bmatrix} \end{bmatrix}$$

$$+ \begin{bmatrix} \sum \left\{ \sum_{s=0}^{1} \left(N_s^T A_{\bar{e}_f} M_s - M_s^T A_{\bar{e}_f} N_s \right) + F^T \left(\mathcal{L}_{U^T \bar{e}_f} + \mathcal{L}_{U^T \bar{e}_f}^T \right) F \right\} & 0 & \sum R^T A_{\bar{e}_f} \\ 0 & 0 & 0 \\ -\sum A_{\bar{e}_f} R & 0 & \sum A_{\bar{e}_f} A_{x_f} + \begin{bmatrix} 0 & -\bar{N}_f & \bar{M}_f \\ 0 & 0 & -\bar{L}_f \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix}$$

...... (A-12)

For convenience we repeat, from Ref 1 (equation (22)) the formula for the datum motion values of the generalised forces, viz:

$$\begin{bmatrix} (\bar{Q}_1)_f \\ \vdots \\ (\bar{Q}_{n+6})_f \end{bmatrix} = \begin{bmatrix} \sum R^T \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} \\ \sum \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} \\ \sum A_{x_f} \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} \end{bmatrix} . \tag{A-13}$$

The additional second order terms (A-8) in the expressions for the deformation (A-7) will of course, make no difference to the first order approximations to the overall forces and moments on the aircraft and so, from Ref 1 (equations (74) to (77))

$$\begin{bmatrix} \bar{X}^{(c)} \\ \bar{Y}^{(c)} \\ \bar{Z}^{(c)} \end{bmatrix} \approx \begin{bmatrix} \bar{X}_f \\ \bar{Y}_f \\ \bar{Z}_f \end{bmatrix} + \sum_{j=1}^{n+6} \begin{bmatrix} \bar{X}_j^{(c)} \\ \bar{Y}_j^{(c)} \\ \bar{Z}_j^{(c)} \end{bmatrix} q_j \tag{A-14}$$

where

$$\begin{bmatrix} \bar{X}_j^{(c)} \\ \bar{Y}_j^{(c)} \\ \bar{Z}_j^{(c)} \end{bmatrix} = \sum \begin{bmatrix} \bar{e}_j \\ \bar{f}_j \\ \bar{g}_j \end{bmatrix} \tag{A-15}$$

and

$$\begin{bmatrix} \bar{L}_c^{(c)} \\ \bar{M}_c^{(c)} \\ \bar{N}_c^{(c)} \end{bmatrix} = \begin{bmatrix} \bar{L}_f \\ \bar{M}_f \\ \bar{N}_f \end{bmatrix} + \sum_{j=1}^{n+6} \begin{bmatrix} \bar{L}_j^{(c)} \\ \bar{M}_j^{(c)} \\ \bar{N}_j^{(c)} \end{bmatrix} q_j \tag{A-16}$$

where

$$\begin{bmatrix} \bar{L}_1^{(c)} & \dots & \bar{L}_{n+6}^{(c)} \\ \bar{M}_1^{(c)} & \dots\dots\dots & \\ \bar{N}_1^{(c)} & \dots\dots\dots & \end{bmatrix} = \sum A_{x_f} \begin{bmatrix} \bar{e}_1 & \dots & \bar{e}_{n+6} \\ \bar{f}_1 & \dots\dots\dots & \\ \bar{g}_1 & \dots\dots\dots & \end{bmatrix} + \left[-\sum A_{\bar{e}_f} R \quad - A_{\bar{X}_f} \quad \sum A_{\bar{e}_f} A_{x_f} \right] . \tag{A-17}$$

We can therefore rewrite (A-12) and (A-13) as

$$\begin{bmatrix} (\bar{Q}_1)_f \\ \vdots \\ (\bar{Q}_{n+6})_f \end{bmatrix} = \begin{bmatrix} \sum R^T \begin{bmatrix} \bar{e}_f \\ \bar{f}_f \\ \bar{g}_f \end{bmatrix} \\ \begin{bmatrix} \bar{X}_f \\ \bar{Y}_f \\ \bar{Z}_f \end{bmatrix} \\ \begin{bmatrix} \bar{L}_f \\ \bar{M}_f \\ \bar{N}_f \end{bmatrix} \end{bmatrix} \tag{A-18}$$

and

$$
[\bar{Q}_{ij}] = \begin{bmatrix}
\sum R^{T}\begin{bmatrix} \bar{e}_1 & \cdots & \bar{e}_{n+6} \\ \bar{f}_1 & \cdots\cdots & \\ \bar{g}_1 & \cdots\cdots & \end{bmatrix} \\
\begin{bmatrix} \bar{X}_1^{(c)} & \cdots & \bar{X}_{n+6}^{(c)} \\ \bar{Y}_1^{(c)} & \cdots\cdots & \\ \bar{Z}_1^{(c)} & \cdots\cdots & \end{bmatrix} \\
\begin{bmatrix} \bar{L}_1^{(c)} & \cdots & \bar{L}_{n+6}^{(c)} \\ \bar{M}_1^{(c)} & \cdots\cdots & \\ \bar{N}_1^{(c)} & \cdots\cdots & \end{bmatrix}
\end{bmatrix}
$$

$$
+ \begin{bmatrix}
\sum\left\{ \sum_{s=0}^{1}\left(N_s^T A_{\bar{e}_f} M_s - M_s^T A_{\bar{e}_f} N_s\right) + F^T\left(\mathcal{L}_{U^T}\begin{bmatrix}\bar{e}_f \\ \bar{f}_f \\ \bar{g}_f\end{bmatrix} + \mathcal{L}^T_{U^T}\begin{bmatrix}\bar{e}_f \\ \bar{f}_f \\ \bar{g}_f\end{bmatrix}\right)F \right\} & 0 & \sum R^T A_{\bar{e}_f} \\
0 & 0 & 0 \\
0 & A_{\bar{X}_f} & \begin{bmatrix} 0 & -\bar{N}_f & \bar{M}_f \\ 0 & 0 & -\bar{L}_f \\ 0 & 0 & 0 \end{bmatrix}
\end{bmatrix}
$$

...... (A-19)

Appendix B

DETERMINATION OF THE LOCATION OF THE PRINCIPAL AXES OF INERTIA

The determination of the location of the principal axes of inertia of the aircraft in the datum state is basically simple but for completeness it was considered worthwhile to restate it here.

Let us take an arbitrary set of body-fixed axes as a first guess at the principal axes of inertia. Having done so we can then divide the aircraft into strips, locate the strip-fixed axes for each strip, and evaluate the quantities x_{if} etc, ϕ_{if} etc, δ_{if} for each strip (*cf* section 3). Note that under our definition the strip-fixed axes are found by moving, in one's imagination, the flap part of the strip until the position is found where the line joining the cg of the strip and the cg of the flap part of the strip passes through the hinge. This line then gives the direction of the x strip-fixed axis. A point is chosen on this line as the origin of these axes (*ie* the strip reference point), and the y axis is 'drawn', as near as possible, normal to both the sides of the strip.

One can then evaluate the quantities (*cf* equations (6-5), (6-6), (6-18) and (6-19))

$$\begin{bmatrix} x_b \\ y_b \\ z_b \end{bmatrix} \text{(say)} = \frac{1}{m} \sum \delta m \begin{bmatrix} x_f \\ y_f \\ z_f \end{bmatrix} \qquad \text{(B-1)}$$

$$I_b \text{ (say)} = -\sum \delta m A_{x_f}^2 + m A_{x_b}^2 \quad . \qquad \text{(B-2)}$$

If our guess of axes had been correct, then $\{x_b \; y_b \; z_b\}$ would be zero and I_b would be diagonal. If this is not so, then we find the axes transformation matrix S_{ϕ_b} which makes $S_{\phi_b} I_b S_{\phi_b}^T$ diagonal. Thus the rows of S_{ϕ_b} are the eigenvectors of I_b normalised so that $S_{\phi_b}^T = S_{\phi_b}^{-1}$. This however leaves some ambiguity - there are six possible matrices S_{ϕ_b} - due to the facts that any of the three principal axes could be called the x-axis, and having done so, there is still the choice to be made as to which is the positive x direction. One suggests therefore making the choice which makes the three diagonal elements of S_{ϕ_b} most nearly unity. The first row and last column of S_{ϕ_b} are as shown below

$$S_{\phi_b} = \begin{bmatrix} \cos\theta_b \cos\psi_b & \cos\theta_b \sin\psi_b & -\sin\theta_b \\ \cdots\cdots & \cdots\cdots & \cos\theta_b \sin\phi_b \\ \cdots\cdots & \cdots\cdots & \cos\theta_b \cos\phi_b \end{bmatrix} \tag{B-3}$$

and so, when S_{ϕ_b} is known, one can, taking the principal values, in general*, uniquely determine the three Euler angles ϕ_b, θ_b, ψ_b . The translation x_b, y_b, z_b , followed by the Euler rotations ψ_b, θ_b, ϕ_b , will then move our guessed axes into coincidence with the principal axes of inertia of the aircraft. Of course, one does not really need to determine the angles ϕ_b etc; the matrix S_{ϕ_b} suffices to give the orientation of the principal axes of inertia.

When the location and orientation of the principal axes of inertia has been determined one repeats the procedure:

(i) Divide the aircraft into strips.

(ii) Locate the strip-fixed axes for each strip.

(iii) Evaluate for each strip $\begin{cases} x_{if} & y_{if} & z_{if} \\ \phi_{if} & \theta_{if} & \psi_{if} \\ \delta_{if} & . & \end{cases}$

* The exception is when $\cos\theta_b = 0$. It then turns out that one can get only either the sum or the difference of ϕ_b and ψ_b . This does not matter of course for this is the case when the carried axis about which the third rotation is made coincides with the original axis about which the first rotation was made.

Appendix C

A BRIEF CONSIDERATION OF THE USE OF STRIP THEORY IN A BODY-FIXED AXES CONTEXT

To use Lagrange's equation for a non-inertial frame one requires that the generalised coordinates for which it is used must not influence directly the position, etc of the non-inertial frame. Taking the non-inertial frame to be the body-fixed axes, and denoting the strip in which the body-fixed axes are fixed as strip 0, we therefore choose generalised coordinates $\hat{q}_1 \rightarrow \hat{q}_n$ for the deformational freedoms such that the aircraft's perturbation from its datum state can be achieved by the following successive steps:

(i) Translations $\hat{x}_1^{(c)}, \hat{y}_1^{(c)}, \hat{z}_1^{(c)}$, in the directions of the constant-velocity axes (also called $\hat{q}_{n+1} \rightarrow \hat{q}_{n+3}$ respectively).

(ii) Euler rotations $\hat{\psi}, \hat{\theta}, \hat{\phi}$ (in that order), about the carried (body-fixed) axes (also called $\hat{q}_{n+6}, \hat{q}_{n+5}, \hat{q}_{n+4}$ respectively).

(iii) Deformations relative to the body-fixed axes such that the position and orientation of the strip-fixed axes of strip i, relative to the body-fixed axes, are given by (*cf* equations (3-2) and (3-3))

$$\begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \begin{bmatrix} x_{if} \\ y_{if} \\ z_{if} \end{bmatrix} + \left\{ \hat{K} - (\hat{K})_0 \right\} \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_n \end{bmatrix} \tag{C-1}$$

$$\begin{bmatrix} \hat{\phi}_i \\ \hat{\theta}_i \\ \hat{\psi}_i \end{bmatrix} = \begin{bmatrix} \phi_{if} \\ \theta_{if} \\ \psi_{if} \end{bmatrix} + \left\{ \hat{F} - (\hat{F})_0 \right\} \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_n \end{bmatrix} . \tag{C-2}$$

In addition, the flap angle perturbation is (*cf* equation (3-7))

$$\delta_i - \delta_{if} = f^T \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_n \end{bmatrix} . \tag{C-3}$$

Consequently, due to a typical force distribution which is such that the overall forces, moments and hinge moment on the ith strip are

$$\begin{bmatrix} \bar{X}_i^{(s)} \\ \bar{Y}_i^{(s)} \\ \bar{Z}_i^{(s)} \end{bmatrix} = \begin{bmatrix} \bar{X}_{if}^{(us)} \\ \bar{Y}_{if}^{(us)} \\ \bar{Z}_{if}^{(us)} \end{bmatrix} + \begin{bmatrix} \bar{\hat{X}}_{i1} & \cdots & \bar{\hat{X}}_{i,n+6} \\ \bar{\hat{Y}}_{i1} & \cdots\cdots & \\ \bar{\hat{Z}}_{i1} & \cdots\cdots & \end{bmatrix} \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix} \tag{C-4}$$

$$\begin{bmatrix} \bar{L}_i^{(s)} \\ \bar{M}_i^{(s)} \\ \bar{N}_i^{(s)} \end{bmatrix} = \begin{bmatrix} \bar{L}_{if}^{(us)} \\ \bar{M}_{if}^{(us)} \\ \bar{N}_{if}^{(us)} \end{bmatrix} + \begin{bmatrix} \bar{\hat{L}}_{i1} & \cdots & \bar{\hat{L}}_{i,n+6} \\ \bar{\hat{M}}_{i1} & \cdots\cdots & \\ \bar{\hat{N}}_{i1} & \cdots\cdots & \end{bmatrix} \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix} \tag{C-5}$$

$$\bar{B}_i = \bar{B}_{if} + \begin{bmatrix} \bar{\hat{B}}_{i1} & \cdots & \bar{\hat{B}}_{i,n+6} \end{bmatrix} \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix} \tag{C-6}$$

the contributions to the generalised forces for the deformational freedoms are, calculated by the principle of virtual work assuming the body-fixed axes stationary.

$$\begin{bmatrix} \bar{\hat{Q}}_1^{(i)} \\ \vdots \\ \bar{\hat{Q}}_n^{(i)} \end{bmatrix} = (\hat{K} - (\hat{K})_0)^T S_{\hat{\phi}_i}^T \begin{bmatrix} \bar{X}_i^{(s)} \\ \bar{Y}_i^{(s)} \\ \bar{Z}_i^{(s)} \end{bmatrix} + (\hat{F} - (\hat{F})_0)^T Q_{\hat{\phi}_i}^T \begin{bmatrix} \bar{L}_i^{(s)} \\ \bar{M}_i^{(s)} \\ \bar{N}_i^{(s)} \end{bmatrix} + f\bar{B}_i \; . \tag{C-7}$$

Now (*cf* equations (3-13), (3-15) and (4-85))

$$S_{\hat{\phi}_i}^T \approx S_{\phi_{if}}^T \left(I + A_{\hat{\alpha}_i} \right) \tag{C-8}$$

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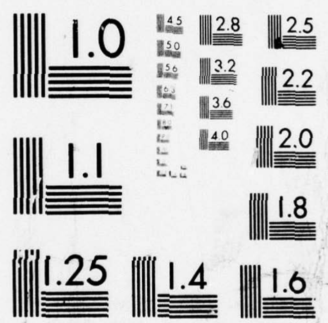

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$$Q^T_{\hat{\phi}_i} \approx Q^T_{\phi_{if}} - \cos\theta_{if}\, J^T \left(\left\{\hat{F} - (\hat{F})_0\right\} \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_n \end{bmatrix} \right) Q^{-1}_{\phi_{if}} \tag{C-9}$$

where
$$\hat{\alpha}_i = Q_{\phi_{if}} \left\{\hat{F} - (\hat{F})_0\right\} \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_n \end{bmatrix} \tag{C-10}$$

and so (C-7) becomes

$$\begin{aligned}
\begin{bmatrix} \bar{\hat{Q}}_1^{(i)} \\ \vdots \\ \bar{\hat{Q}}_n^{(i)} \end{bmatrix} \approx\; & (\hat{K} - (\hat{K})_0)^T S^T_{\phi_{if}} \begin{bmatrix} \bar{X}^{(us)}_{if} \\ \bar{Y}^{(us)}_{if} \\ \bar{Z}^{(us)}_{if} \end{bmatrix} + (\hat{F} - (\hat{F})_0)^T Q^T_{\phi_{if}} \begin{bmatrix} \bar{L}^{(us)}_{if} \\ \bar{M}^{(us)}_{if} \\ \bar{N}^{(us)}_{if} \end{bmatrix} + f\bar{B}_{if} \\
& + \left\{ (\hat{K} - (\hat{K})_0)^T S^T_{\phi_{if}} \left[\begin{bmatrix} \bar{\hat{X}}_{i1} & \ldots & \bar{\hat{X}}_{in} \\ \bar{\hat{Y}}_{i1} & \ldots\ldots & \\ \bar{\hat{Z}}_{i1} & \ldots\ldots & \end{bmatrix} - A_{\bar{X}^{(us)}_{if}} Q_{\phi_{if}} (\hat{F} - (\hat{F})_0) \begin{bmatrix} \bar{\hat{X}}_{i,n+1} & \ldots & \bar{\hat{X}}_{i,n+6} \\ \bar{\hat{Y}}_{i,n+1} & \ldots\ldots & \\ \bar{\hat{Z}}_{i,n+1} & \ldots\ldots & \end{bmatrix} \right] \right. \\
& + (\hat{F} - (\hat{F})_0)^T Q^T_{\phi_{if}} \left[\begin{bmatrix} \bar{\hat{L}}_{i1} & \ldots & \bar{\hat{L}}_{in} \\ \bar{\hat{M}}_{i1} & \ldots\ldots & \\ \bar{\hat{N}}_{i1} & \ldots\ldots & \end{bmatrix} - \cos\theta_{if}\, \mathcal{L} \left(Q^{-1}_{\phi_{if}} \begin{bmatrix} \bar{L}^{(us)}_{isf} \\ \bar{M}^{(us)}_{isf} \\ \bar{N}^{(us)}_{isf} \end{bmatrix} \right) (\hat{F} - (\hat{F})_0) \begin{bmatrix} \bar{\hat{L}}_{i,n+1} & \ldots & \bar{\hat{L}}_{i,n+6} \\ \bar{\hat{M}}_{i,n+1} & \ldots\ldots & \\ \bar{\hat{N}}_{i,n+1} & \ldots\ldots & \end{bmatrix} \right] \\
& \left. + f \begin{bmatrix} \bar{\hat{B}}_{i1} & \ldots & \bar{\hat{B}}_{i,n+6} \end{bmatrix} \right\} \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix}
\end{aligned}$$

...... (C-11)

Lagrange's equations for a non-inertial frame, taking the non-inertial frame to be the body-fixed axes, will only provide us with the n equations for the deformational freedoms. The other six equations are, as in Ref 1, obtained from consideration of the rates of change of the linear momentum, and angular momentum about the aircraft reference point, resolved along the body-fixed axes. Thus we require the resolutes along the body-fixed axes of the total applied force and moment about the reference point. The contributions to these from a typical force distribution and from the ith strip are (*cf* equations (C-4) and (C-5))

$$\begin{bmatrix}\bar{\hat{X}}_i\\ \bar{\hat{Y}}_i\\ \bar{\hat{Z}}_i\end{bmatrix} = S^T_{\phi_i}\begin{bmatrix}\bar{X}_i^{(s)}\\ \bar{Y}_i^{(s)}\\ \bar{Z}_i^{(s)}\end{bmatrix}$$

$$\approx S^T_{\phi_{if}}\left\{\begin{bmatrix}\bar{X}_{if}^{(us)}\\ \bar{Y}_{if}^{(us)}\\ \bar{Z}_{if}^{(us)}\end{bmatrix} + \left[\begin{bmatrix}\bar{\hat{X}}_{i1} & \cdots & \bar{\hat{X}}_{in}\\ \bar{\hat{Y}}_{i1} & \cdots\cdots & \\ \bar{\hat{Z}}_{i1} & \cdots\cdots & \end{bmatrix} - A_{\bar{X}_{if}^{(us)}} Q_{\phi_{if}} (\hat{F} - (\hat{F})_0) \begin{bmatrix}\bar{\hat{X}}_{i,n+1} & \cdots & \bar{\hat{X}}_{i,n+6}\\ \bar{\hat{Y}}_{i,n+1} & \cdots\cdots & \\ \bar{\hat{Z}}_{i,n+1} & \cdots\cdots & \end{bmatrix}\right]\begin{bmatrix}\hat{q}_1\\ \vdots\\ \hat{q}_{n+6}\end{bmatrix}\right\} \qquad \ldots\ldots \text{(C-12)}$$

$$\begin{bmatrix}\bar{\hat{L}}_i\\ \bar{\hat{M}}_i\\ \bar{\hat{N}}_i\end{bmatrix} = S^T_{\phi_i}\begin{bmatrix}\bar{L}_i^{(s)}\\ \bar{M}_i^{(s)}\\ \bar{N}_i^{(s)}\end{bmatrix} + A_{x_i} S^T_{\phi_i}\begin{bmatrix}\bar{X}_i^{(s)}\\ \bar{Y}_i^{(s)}\\ \bar{Z}_i^{(s)}\end{bmatrix}$$

$$\approx S^T_{\phi_{if}}\begin{bmatrix}\bar{L}_{if}^{(us)}\\ \bar{M}_{if}^{(us)}\\ \bar{N}_{if}^{(us)}\end{bmatrix} + A_{x_{if}} S^T_{\phi_{if}}\begin{bmatrix}\bar{X}_{if}^{(us)}\\ \bar{Y}_{if}^{(us)}\\ \bar{Z}_{if}^{(us)}\end{bmatrix}$$

$$+ S^T_{\phi_{if}}\left[\begin{bmatrix}\bar{\hat{L}}_{i1} & \cdots & \bar{\hat{L}}_{in}\\ \bar{\hat{M}}_{i1} & \cdots\cdots & \\ \bar{\hat{N}}_{i1} & \cdots\cdots & \end{bmatrix} + S_{\phi_{if}} A_{x_{if}} S^T_{\phi_{if}}\begin{bmatrix}\bar{\hat{X}}_{i1} & \cdots & \bar{\hat{X}}_{in}\\ \bar{\hat{Y}}_{i1} & \cdots\cdots & \\ \bar{\hat{Z}}_{i1} & \cdots\cdots & \end{bmatrix} - A_{\bar{L}_{if}^{(us)}} Q_{\phi_{if}} (\hat{F} - (\hat{F})_0)\right.$$

$$- S_{\phi_{if}} A_{x_{if}} S^T_{\phi_{if}} A_{\bar{X}_{if}^{(us)}} Q_{\phi_{if}} (\hat{F} - (\hat{F})_0) - A_{\bar{X}_{if}^{(us)}} S_{\phi_{if}} (\hat{R} - (\hat{R})_0)$$

$$\left.\begin{bmatrix}\bar{\hat{L}}_{i,n+1} & \cdots & \bar{\hat{L}}_{i,n+6}\\ \bar{\hat{M}}_{i,n+1} & \cdots\cdots & \\ \bar{\hat{N}}_{i,n+1} & \cdots\cdots & \end{bmatrix} + S_{\phi_{if}} A_{x_{if}} S^T_{\phi_{if}}\begin{bmatrix}\bar{\hat{X}}_{i,n+1} & \cdots & \bar{\hat{X}}_{i,n+6}\\ \bar{\hat{Y}}_{i,n+1} & \cdots\cdots & \\ \bar{\hat{Z}}_{i,n+1} & \cdots\cdots & \end{bmatrix}\right]\begin{bmatrix}\hat{q}_1\\ \vdots\\ \hat{q}_{n+6}\end{bmatrix} \qquad \ldots\ldots \text{(C-13)}$$

It is not as easy to relate the derivation of this Appendix to that for constant-velocity axes (in the main part of this paper) as it was in Ref 1 when the deformation was not written in a form attractive for strip theory.

The reader will have noted that in equations (C-1) and (C-2), we could not put $\{K - (K)_0\}$ and $\{F - (F)_0\}$, where K and F represented the same modal functions as in the main part of the paper, but had to add the distinguishing circumflexes. In particular it will be appreciated that the term $(F)_0 \{\hat{q}_1 \dots \hat{q}_n\}$, in (C-2), does not represent a rigid body rotation, not even to a first approximation. It is an approximately constant rotation about axes which vary from strip to strip, or in other words it is, about one set of axes, the rotation

$$S^T_{\phi_{if}} Q_{\phi_{if}} (\hat{F})_0 \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_n \end{bmatrix} + \text{higher order terms} \tag{C-14}$$

which varies from strip to strip (*cf* equations (3-13) and (3-15)). From equation (4-7) it will be seen that this rotation is approximately constant if θ_{if} is the same for all strips.

If the deformation of section 3 is associated with just sufficient rigid body motion to make the displacement and slope, at the aircraft reference point, due to the deformation, zero, it means that the rigid body displacements are:

rotations ϕ satisfying*

$$S_\phi = S^T_{\phi_0} S_{\phi_{0f}} \tag{C-15}$$

and translations*

$$\begin{bmatrix} x_1^{(c)} \\ y_1^{(c)} \\ z_1^{(c)} \end{bmatrix} = - S^T_{\phi_{0f}} S_{\phi_0} \left\{ (K)_0 \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} + \left(I - S^T_{\phi_0} S_{\phi_{0f}}\right) \begin{bmatrix} x_{0f} \\ y_{0f} \\ z_{0f} \end{bmatrix} \right\} , \tag{C-16}$$

---

* This means there is no change in orientation or position in going from the unperturbed strip-fixed axes to the strip-fixed axes, for strip 0 (*cf* equations (4-38) and (4-39)).

ie

$$\phi \approx -P_q \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \tag{C-17}$$

and

$$x_1^{(c)} \approx -\left\{(K)_0 + A_{x_{0f}} P_q\right\} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \tag{C-18}$$

where P_q is given by equation (5-55).

Thus, if we take

$$\hat{F} - (\hat{F})_0 = F - Q_{\phi_{if}}^{-1} S_{\phi_{if}} P_q = F - Q_{\phi_{if}}^{-1} S_{\phi_{if}} S_{\phi_{0f}}^T Q_{\phi_{0f}} (F)_0 \tag{C-19}$$

$$\hat{K} - (\hat{K})_0 = K - (K)_0 - \left(A_{x_{0f}} - A_{x_{if}}\right) P_q \tag{C-20}$$

which for arbitrary F , K can be achieved by taking

$$\hat{F} = F - Q_{\phi_{if}}^{-1} S_{\phi_{if}} S_{\phi_{0f}}^T Q_{\phi_{0f}} (F)_0 \tag{C-21}$$

(and so, as expected, taking $(\hat{F})_0 = 0$), and

$$\hat{K} = K + A_{x_{if}} P_q \tag{C-22}$$

then, to a first approximation, the modes of deformation relative to the strip-fixed axes will be the same with both representations (that of section 3 and that of this Appendix). This suggests that with the modal relationships given above the following relationship might not be far from the truth

$$\begin{bmatrix} \bar{\bar{Q}}_1^{(i)} \\ \vdots \\ \bar{\bar{Q}}_n^{(i)} \end{bmatrix} = \begin{bmatrix} I & -\left\{(K)_0 + A_{x_{0f}} P_q\right\}^T & -P_q^T \end{bmatrix} \begin{bmatrix} \bar{Q}_1^{(i)} \\ \vdots \\ \bar{Q}_{n+6}^{(i)} \end{bmatrix} . \tag{C-23}$$

From (5-11) and (C-11) it is easily seen to be true for the datum motion values of these generalised forces. For both models to represent, to a good approximation, the same perturbation there will be a similar relationship, corresponding to (C-23), between the generalised coordinates, which we can deduce to be

$$\begin{bmatrix} q_1 \\ \vdots \\ q_{n+6} \end{bmatrix} \approx \begin{bmatrix} I & 0 & 0 \\ -\left\{(K)_0 + A_{x_{0f}} P_q\right\} & I & 0 \\ -P_q & 0 & I \end{bmatrix} \begin{bmatrix} \hat{q}_1 \\ \vdots \\ \hat{q}_{n+6} \end{bmatrix} . \qquad \text{(C-24)}$$

Comparing the expressions for the overall forces moments and hinge moment on a strip - equations (C-4) to (C-6) with equations (5-7) to (5-9) we then immediately obtain the relationships

$$\begin{bmatrix} \bar{\hat{X}}_{i1} & \cdots & \bar{\hat{X}}_{i,n+6} \\ \bar{\hat{Y}}_{i1} & \cdots\cdots\cdots & \\ \bar{\hat{Z}}_{i1} & \cdots\cdots\cdots & \end{bmatrix} = \begin{bmatrix} \bar{X}_{i1} & \cdots & \bar{X}_{i,n+6} \\ \bar{Y}_{i1} & \cdots\cdots\cdots & \\ \bar{Z}_{i1} & \cdots\cdots\cdots & \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ -\left\{(K)_0 + A_{x_{0f}} P_q\right\} & I & 0 \\ -P_q & 0 & I \end{bmatrix}$$

$$\ldots\ldots \qquad \text{(C-25)}$$

etc, when the modal functions are related by equations (C-19) and (C-20). Substituting these relationships into equations (C-12) and (C-13), and writing

$$\begin{bmatrix} \bar{\hat{X}}_i \\ \bar{\hat{Y}}_i \\ \bar{\hat{Z}}_i \\ \bar{\hat{L}}_i \\ \bar{\hat{M}}_i \\ \bar{\hat{N}}_i \end{bmatrix} = \begin{bmatrix} \bar{\hat{Q}}^{(i)}_{n+1} \\ \vdots \\ \bar{\hat{Q}}^{(i)}_{n+6} \end{bmatrix} \approx \begin{bmatrix} \bar{\hat{Q}}_{n+1,f} \\ \vdots \\ \bar{\hat{Q}}_{n+6,f} \end{bmatrix} + \sum_{k=1}^{n+6} \begin{bmatrix} \bar{\hat{Q}}^{(i)}_{n+1,k} \\ \\ \bar{\hat{Q}}_{n+6,k} \end{bmatrix} q_k \qquad \text{(C-26)}$$

we then find, after some analysis, that (*cf* equations (5-11))

$$\begin{bmatrix} \bar{\hat{Q}}_{n+1,f} \\ \vdots \\ \bar{\hat{Q}}_{n+6,f} \end{bmatrix} = \begin{bmatrix} \bar{Q}^{(i)}_{n+1,f} \\ \vdots \\ \bar{Q}^{(i)}_{n+6,f} \end{bmatrix} \tag{C-27}$$

and

$$\begin{bmatrix} \bar{\hat{Q}}^{(i)}_{n+1,1} & \cdots & \bar{\hat{Q}}^{(i)}_{n+1,n+6} \\ \cdots & \cdots & \cdots \\ \bar{\hat{Q}}^{(i)}_{n+6,1} & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \bar{Q}^{(i)}_{jk} \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ -\left\{(K)_0 + A_{x_{0f}} P_q\right\} & I & 0 \\ -P_q & 0 & I \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & S^T_{\phi_{if}} A_{\bar{X}^{(us)}_{if}} S_{\phi_{if}} \\ S^T_{\phi_{if}} A_{\bar{X}^{(us)}_{if}} S_{\phi_{if}} \left\{(K)_0 + A_{x_{0f}} P_q\right\} + \mathcal{L}^T \left(A_{x_{if}} S^T_{\phi_{if}} \begin{bmatrix} \bar{X}^{(us)}_{if} \\ \bar{Y}^{(us)}_{if} \\ \bar{Z}^{(if)}_{if} \end{bmatrix} + S^T_{\phi_{if}} \begin{bmatrix} \bar{L}^{(us)}_{if} \\ \bar{M}^{(us)}_{if} \\ \bar{N}^{(us)}_{if} \end{bmatrix} \right) P_q & 0 & \mathcal{L} \left(A_{x_{if}} S^T_{\phi_{if}} \begin{bmatrix} \bar{X}^{(us)}_{if} \\ \bar{Y}^{(us)}_{if} \\ \bar{Z}^{(us)}_{if} \end{bmatrix} + S^T_{\phi_{if}} \begin{bmatrix} \bar{L}^{(us)}_{if} \\ \bar{M}^{(us)}_{if} \\ \bar{N}^{(us)}_{if} \end{bmatrix} \right) \end{bmatrix}$$

$$\ldots\ldots \quad \text{(C-28)}$$

This is exactly the same relationship as that obtained in Ref 1 (equation (161)). The relationship of Ref 1 also holds for those coefficients which express the influence of the body freedoms on the deformational freedoms, but not, as one would expect because of the differences displayed in Appendix A, for the 'deformation-deformation' coefficients. Comparing (C-11) and (5-12) we find that, for the modal relationships of (C-19) and (C-20),

$$
\begin{bmatrix} \bar{\bar{Q}}^{(i)}_{11} & \cdots & \bar{\bar{Q}}^{(i)}_{1,n+6} \\ \cdots & \cdots & \cdots \\ \bar{\bar{Q}}^{(i)}_{n1} & \cdots & \cdots \end{bmatrix} = \begin{bmatrix} I & -\left\{(K)_0 + A_{x_{0f}} P_q\right\}^T & -P_q^T \end{bmatrix} \times
$$

$$
\times \left\{ \begin{bmatrix} \bar{\bar{Q}}^{(i)}_{jk} \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ -\left\{(K)_0 + A_{x_{0f}} P_q\right\} & I & 0 \\ -P_q & 0 & I \end{bmatrix} + \right.
$$

$$
\left. + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & S^T_{\phi_{if}} A_{\bar{X}^{(us)}_{if}} S_{\phi_{if}} \\ S^T_{\phi_{if}} A_{\bar{X}^{(us)}_{if}} S_{\phi_{if}} \left\{(K)_0 + A_{x_{0f}} P_q\right\} + \mathcal{L}^T \left(A_{x_{if}} S^T_{\phi_{if}} \begin{bmatrix} \bar{X}^{(us)}_{if} \\ \bar{Y}^{(uc)}_{if} \\ \bar{Z}^{(us)}_{if} \end{bmatrix} + S^T_{\phi_{if}} \begin{bmatrix} \bar{L}^{(us)}_{if} \\ \bar{M}^{(us)}_{if} \\ \bar{N}^{(us)}_{if} \end{bmatrix} \right) P_q & 0 & \mathcal{L} \left(A_{x_{if}} S^T_{\phi_{if}} \begin{bmatrix} \bar{X}^{(us)}_{if} \\ \bar{Y}^{(us)}_{if} \\ \bar{Z}^{(us)}_{if} \end{bmatrix} + S^T_{\phi_{if}} \begin{bmatrix} \bar{L}^{(us)}_{if} \\ \bar{M}^{(us)}_{if} \\ \bar{N}^{(us)}_{if} \end{bmatrix} \right) \end{bmatrix} \right\} + \begin{bmatrix} \delta & 0 & 0 \end{bmatrix}
$$

...... (C-29)

where

$$
\delta = P_q^T S^T_{\phi_{if}} A_{\bar{L}^{(us)}_{if}} S_{\phi_{if}} P_q + K^T S^T_{\phi_{if}} A_{\bar{X}^{(us)}_{if}} S_{\phi_{if}} P_q - P_q^T S^T_{\phi_{if}} A_{\bar{X}^{(us)}_{if}} S_{\phi_{if}} \left\{ \left(K + A_{x_{if}} P_q \right) - \left((K)_0 + A_{x_{0f}} P_q \right) \right\}
$$

$$
+ \cos\theta_{if} \left\{ F^T \mathcal{L} \left(Q^{-1}_{\phi_{if}} \begin{bmatrix} \bar{L}^{(us)}_{if} \\ \bar{M}^{(us)}_{if} \\ \bar{N}^{(us)}_{if} \end{bmatrix} \right) Q^{-1}_{\phi_{if}} S_{\phi_{if}} P_q + P_q^T S^T_{\phi_{if}} \left(Q^{-1}_{\phi_{if}} \right)^T \mathcal{L}^T \left(Q^{-1}_{\phi_{if}} \begin{bmatrix} \bar{L}^{(us)}_{if} \\ \bar{M}^{(us)}_{if} \\ \bar{N}^{(us)}_{if} \end{bmatrix} \right) F \right.
$$

$$
\left. - P_q^T S^T_{\phi_{if}} \left(Q^{-1}_{\phi_{if}} \right)^T \mathcal{L} \left(Q^{-1}_{\phi_{if}} \begin{bmatrix} \bar{L}^{(us)}_{if} \\ \bar{M}^{(us)}_{if} \\ \bar{N}^{(us)}_{if} \end{bmatrix} \right) Q^{-1}_{\phi_{if}} S_{\phi_{if}} P_q \right\} .
$$

...... (C-30)

In Ref 1, because of the slightly different model, the expression for $\&$ is different and is $\& = \&_{77}$ where (*cf* equation (161) of Ref 1)

$$
\begin{aligned}
\&_{77} = {} & P_q^T S^T_{\phi_{if}} A^{(us)}_{\bar{X}_{if}} S_{\phi_{if}} \left((K)_0 + A_{x_{0f}} P_q \right) \\
& + \sum \Big\{ R^T S^T_{\phi_{if}} A^{(us)}_{\bar{e}_{if}} S_{\phi_{if}} P_q - P_q^T S^T_{\phi_{if}} A^{(us)}_{\bar{e}_{if}} S_{\phi_{if}} R \\
& \qquad - P_q^T S^T_{\phi_{if}} A^{(us)}_{\bar{e}_{if}} S_{\phi_{if}} \left(A_{x_{if}} + S^T_{\phi_{if}} A^{(us)}_{x_{sif}} S_{\phi_{if}} \right) P_q \Big\}
\end{aligned}
\qquad \text{...... (C-31)}
$$

the summation is over the ith strip, and R is given by equation (3-20). One term is clearly common to (C-30) and (C-31).

GLOSSARY OF TERMS

(i) Frames of reference (all right-handed orthogonal cartesian)

body-fixed axes	axes whose origin and orientation are fixed in a small material portion of the body. (The ones used are such that the origin is in the main part of strip 0 during the datum motion.)*
constant-velocity axes	axes having constant linear and angular velocity relative to an inertial frame. (The ones used have zero angular velocity and are taken to be coincident with the datum-attitude earth axes during the datum motion.)
datum-attitude earth axes	axes with which the body-fixed axes coincide during the datum motion
datum-path earth axes	axes with the x-axis in the direction of the datum velocity, the xz plane vertical, and the z axis downwards
no-deformation-body-fixed axes	an axes system, arbitrary except insofar as it is of the same order of nearness to the body-fixed axes and the datum-attitude earth axes. (These three frames of reference are all assumed to remain close together during the perturbed motion.)
normal earth-fixed axes	axes fixed relative to the earth with the z-axis vertically downwards
principal axes of inertia	axes with origin at the centre of gravity of the body and such that the three products of inertia about the axes are zero
strip-fixed axes	axes fixed in a strip of the aircraft such that their origin is at the strip reference point the x-axis passes through a point on the strip hinge (and in the unperturbed state, the strip cg), the y-axis is approximately normal to the planar sides of the strip,

* Whether the body-fixed axes remain orthogonal, during perturbations of the datum motion, or not, is irrelevant to the present development.

GLOSSARY OF TERMS (continued)

strip-fixed axes (continued)	and the positive x-direction is towards the strip leading edge - *cf* Appendix B. (For a strip without a hinge one would take the x-axis to be some chord line and the y-axis to be, as near as possible, normal to the planes dividing the strip from the adjacent strips.)
unperturbed-strip-fixed axes	axes whose position coincides with the position the strip-fixed axes would occupy if there was no perturbation from the datum motion.

(ii) Orientation

(a) Attitude angles defining the attitude of the aircraft relative to the normal earth-fixed axes.

angle of bank	angle between the z-axis, of the body-fixed axes and the vertical plane containing the x-axis of the same frame
angle of inclination	angle between the x-axis of the body-fixed axes and the horizontal plane
nose-azimuth angle	angle between the projection of the x-axis, of the body-fixed axes, on the horizontal plane, and the x-axis of the normal earth-fixed axes.

(b) Flight path angle defining the direction of flight relative to the normal earth-fixed axes.

angle of climb	angle between the tangent to the flight path and the horizontal plane, positive when climbing
angle of track	angle between the x-axis of the normal earth-fixed axes and the projection on the horizontal plane of the tangent to the flight path, positive when a right-handed rotation about the downward vertical takes one from the normal earth-fixed x-axis to the flight path projection.

GLOSSARY OF TERMS (concluded)

(c) Incidence angles defining the direction of flight relative to the air

incidence magnitude*	magnitude of angle between the x-axis of the body-fixed axes and the direction of the velocity of the aircraft relative to the air
incidence-plane angle*	angle between the incidence plane and the zx plane of the body-fixed axes.

(iii) Miscellaneous

aircraft reference point	the material point on the aircraft which is the origin of the body-fixed axes
attitude deviation matrix	the matrix which by premultiplying a column vector changes the axes directions in respect to which the vector is resolved into components
axes transformation matrix	synonymous with attitude deviation matrix
incidence plane	the plane defined by the x-axis of the body-fixed axes and the direction of the velocity of the aircraft relative to the air
reference axis	a line, not necessarily straight, joining the strip reference points of the strips of a component, such as the wing, of the aircraft
reference point	synonymous with aircraft reference point
strip reference point	a chosen material point on the main part of a strip taken as the origin of the strip-fixed axes.

* These definitions can similarly be used for a strip or aerofoil section, in terms of the strip-fixed axes.

LIST OF SYMBOLS

A_ϕ etc	skew-symmetric matrices involving ϕ, θ, ψ, etc (see equation (3-16))
A_{jk}	ponderous inertia coefficient
B_i	aerodynamic hinge moment on the ith strip
$B_{i\dot{x}}, B_{i\dot{\theta}}$ etc	aerodynamic hinge moment coefficients for ith strip
$B_{ixj}, B_{i\dot{x}j}$ etc	aerodynamic hinge moment coefficients of, say, a velocity of strip j, in the expression for the hinge moment on strip i
$B^*_{i\theta j}$	a modified $B^{(2)}_{i\theta j}$ - see equation (4-146)
$B_{ijx}, B_{ij\theta}$ etc	aerodynamic hinge moment coefficients expressing the effect, on the hinge moment on strip i, of a change in relative position and orientation of strips i and j
D	differential operator d/dt
$-E_j$	generalised structural force
E_{jk}	structural stiffness coefficient
F	torsional modal matrix (see equation (3-3))
$-G_j$	generalised gravitational force
G_{jk}	gravitational stiffness coefficient
H	Heaviside step function
I	unit matrix
I_x, I_y, I_z	principal moments of inertia of undeformed aircraft
I_n	matrix of moments and products of inertia of undeformed aircraft (see equation (6-6))
I_{i0}	matrix of moments and products of inertia of the ith strip (see equation (6-8))
I_{i1}	matrix of moments and products of inertia of the flap of the ith strip (see equation (6-9))
J_ϕ etc	matrices formed from the elements of $\{\phi\ \theta\ \psi\}$ etc, see equation (4-88)
K	flexural modal matrix (see equation (3-2))
L	aerodynamic rolling moment
$\bar{L}$	typical rolling moment
L_g, L_p	gravitational, propulsive rolling moments
L_i, L_{gi} etc	rolling moments on ith strip
M	aerodynamic pitching moment
$\bar{M}$	typical pitching moment

LIST OF SYMBOLS (continued)

M_0, M_1	modal matrices, see Appendix A
M_g, M_p	gravitational, propulsive pitching moments
M_i, M_{gi} etc	pitching moments on ith strip
$M_{i\dot{x}}, M_{i\dot{\theta}}$ etc	aerodynamic pitching moment coefficients for ith strip
$M_{i\dot{x}j}, M_{i\dot{z}j}$ etc	aerodynamic pitching moment coefficients of, say, a velocity of strip j, in the expression for the pitching moment on strip i
M^*_{ixj}, M^*_{izj}	modified M_{ixj} etc - see equations (4-140) to (4-142)
$M_{ijx}, M_{ij\theta}$ etc	aerodynamic pitching moment coefficients expressing the effect, on the pitching moment on strip i, of a change in the relative position and orientation of strips i and j
N	aerodynamic yawing moment
$\bar{N}$	typical yawing moment
N_0, N_1	modal matrices, see Appendix A
N_g, N_p	gravitational, propulsive yawing moments
N_i, N_{gi} etc	yawing moments on ith strip
$-P_j$	generalised propulsive force
P_q	matrix of modal slopes at reference point (see equations (5-54) and (5-55))
P_θ	matrix which is 'θ factor' of axes transformation matrix (see equation (3-11))
P_{jk}	propulsive stiffness coefficient
Q_j	generalised aerodynamic force
$\bar{Q}_j$	total generalised force, or typical contribution to generalised force
Q_ϕ	matrix relating angular velocities and orientation (see equations (3-14) and (4-3))
$Q_j^{(i)}$	contribution to generalised aerodynamic force from strip i
$\bar{Q}_j^{(i)}$	typical contribution to generalised force from strip i
Q_{jk}	aerodynamic coefficient
R	modal matrix (see equation (3-20))
R_ϕ	matrix which is 'ϕ factor' of axes transformation matrix (see equation (3-10))

LIST OF SYMBOLS (continued)

S, S_{ϕ_i}	axes transformation matrix (see section 3). The subscript is added when it is necessary to specify the arguments - in this case ϕ_i, θ_i, ψ_i . Absence of a subscript means the arguments are ϕ, θ, ψ
U	modal matrix (see Appendix A)
V	airspeed
W	kinetic energy relative to the constant-velocity axes
X	overall aerodynamic force resolute
$\bar{X}$	typical overall force resolute
X_g, X_p	gravitational, propulsive overall force resolutes
X_i, X_{gi} etc	overall force resoluted on strip i
$X_{i\dot{x}}, X_{i\dot{\theta}}$ etc	aerodynamic force resolute coefficients for ith strip
$X_{i\dot{x}j}, X_{i\dot{z}j}$ etc	aerodynamic force resolute coefficients of, say, a velocity of strip j , in the expression for the overall force on strip i
$X^*_{i\theta j}$	modified $X_{i\theta j}$ - see equations (4-128) and (4-129)
$X_{ijx}, X_{ij\theta}$ etc	aerodynamic force resolute coefficients expressing the effect, on the overall force on strip i , of a change in the relative position and orientation of strips i and j
Y	overall aerodynamic force resolute
$\bar{Y}$	typical overall force resolute
Y_g, Y_p	gravitational, propulsive overall force resolutes
Y_ψ	matrix which is 'ψ factor' of axes transformation matrix (see equation (3-12))
Y_i, Y_{gi} etc	overall force resolutes on strip i
Z	overall aerodynamic force resolute
$\bar{Z}$	typical overall force resolute
Z_g, Z_p	gravitational, propulsive overall force resolutes
Z_i, Z_{gi} etc	overall force resolutes on strip i
$Z_{i\dot{x}}, Z_{i\dot{\theta}}$ etc	aerodynamic force resolute coefficients for the ith strip
$Z_{i\dot{x}j}, Z_{i\dot{z}j}$ etc	aerodynamic force resolute coefficients of, say, a velocity of strip j , in the expression for the overall force on strip i
$Z^*_{i\theta j}$	modified $Z_{i\theta j}$ - see equations (4-130) and (4-131)

LIST OF SYMBOLS (continued)

a_{i0}, a_{i1}, a_{i3}	certain column vectors - see equations (5-19) to (5-21)
e_i, e_{gi}	x-component of local aerodynamic, gravitational force vector at ith strip
f	flap modal vector (see equation (3-7))
f_i, f_{gi}	y-component of local aerodynamic, gravitational force vector at ith strip
g	acceleration due to gravity
g_i, g_{gi}	z-component of local aerodynamic, gravitational force vector at ith strip
ℓ_{Φ_f}	third column of S_{Φ_f} (see equation (5-14))
$\ell_\alpha, \ell_z, \ell_{\dot{z}}$	aerodynamic derivatives - obsolescent notation (see section 4.22)
m	mass of aircraft
m_i	mass of strip i of the aircraft
δm	mass of a particle
n	number of deformational degrees of freedom
p	angular velocity resolute
p_i	angular velocity resolute of ith strip
q	angular velocity resolute
q_i	angular velocity resolute of ith strip
q_r	generalised coordinate
r	angular velocity resolute
r_i	angular velocity resolute of ith strip
t	time
u	linear velocity resolute
u_i	linear velocity of ith strip reference point
u_{mi}	particle velocity resolute at ith strip
v	linear velocity resolute
v_i	linear velocity resolute of ith strip reference point
v_{mi}	particle velocity resolute at ith strip
w	linear velocity resolute
w_i	linear velocity resolute of ith strip reference point
w_{mi}	particle velocity resolute at ith strip

LIST OF SYMBOLS (continued)

x	particle position resolute
x_1 ($\equiv q_{n+1}$)	resolute of reference point position, relative to its position in datum motion, excluding deformational contribution
x_i	resolute of strip reference point position
x_{i0}	x-coordinate of strip cg, referred to strip-fixed axes, when the flap angle is zero
x_{i2}	x-coordinate of strip cg, in datum motion, in body-fixed axes frame of reference
x_{hsi}	x-coordinate of strip hinge, referred to strip-fixed axes
x_f^{ij}	see equation (4-109)
y	particle position resolute
y_1 ($\equiv q_{n+2}$)	resolute of reference point position, relative to its position in datum motion, excluding the deformational contribution
y_i	resolute of strip reference point position
y_{if0}	y-coordinate of strip cg, in datum motion, in body-fixed axes frame of reference
z	particle position resolute
z_1 ($\equiv q_{n+3}$)	resolute of reference point position, relative to its position in datum motion, excluding the deformational contribution
z_{if0}	z-coordinate of strip cg, in datum motion, in body-fixed axes frame of reference
z_f^{ij}	see equation (4-110)
E_{qq}	submatrix of structural stiffness coefficients
$G_{qq}, G_{q\phi}, G_{\phi q}, G_{\phi\phi}$	submatrices of gravitational stiffness coefficients
$L_{cq}, L_{cx}, L_{c\phi}$	submatrices of coefficients in expression for aerodynamic rolling moment about origin of constant-velocity axes
$L_{cpq}, L_{cpx}, L_{cp\phi}$	submatrices of coefficients in expression for propulsive rolling moment about origin of constant-velocity axes
$M_{iq}, M_{ix}, M_{i\phi}$	submatrices of coefficients in expression for aerodynamic pitching moment on ith strip
$M_{cq}, M_{cx}, M_{c\phi}$	submatrices of coefficients in expression for aerodynamic pitching moment about origin of constant-velocity axes
$M_{cpq}, M_{cpx}, M_{cp\phi}$	submatrices of coefficients in expression for propulsive pitching moment about origin of constant-velocity axes
$N_{cq}, N_{cx}, N_{c\phi}$	submatrices of coefficients in expression for aerodynamic yawing moment about origin of constant-velocity axes

LIST OF SYMBOLS (continued)

$N_{cpq}, N_{cpx}, N_{cp\phi}$	submatrices of coefficients in expression for propulsive yawing moment about origin of constant-velocity axes
$P_{qq}, P_{xq}, P_{\phi q}$, $P_{x\phi}, P_{\phi\phi}$	submatrices of propulsive stiffness coefficients
X_q, X_x, X_ϕ	submatrices of coefficients in expression for overall aerodynamic force resolute
$X_{iq}, X_{ix}, X_{i\phi}$	submatrices of coefficients in expression for overall aerodynamic force resolute on ith strip
$X_{pq}, X_{px}, X_{p\phi}$	submatrices of coefficients in expression for overall propulsive force resolute
Y_q, Y_x, Y_ϕ	submatrices of coefficients in expression for overall aerodynamic force resolute
$Y_{pq}, Y_{px}, Y_{p\phi}$	submatrices of coefficients in expression for overall propulsive force resolute
Z_q, Z_x, Z_ϕ	submatrices of coefficients in expression for overall aerodynamic force resolute
$Z_{iq}, Z_{ix}, Z_{i\phi}$	submatrices of coefficients in expression for overall aerodynamic force resolute on ith strip
$Z_{pq}, Z_{px}, Z_{p\phi}$	submatrices of coefficients in expression for overall propulsive force resolute
Γ	vortex strength
$\Gamma_u, \Gamma_w, \Gamma_q, \Gamma_f$	coefficients in expression for vortex strength (*cf* equation (4-13))
$\Delta x, \Delta z, \Delta\theta$	see equations (4-106) to (4-108)
Θ	angle of inclination
Φ	angle of bank
Ψ	nose-azimuth angle
α	incidence of aerofoil
α_0	no lift incidence of aerofoil
α_1	no pitching moment incidence for aerofoil - pitching moment is about centre of the circle from which the profile is generated by conformal transformation
α_i	first element of column vector given by equation (3-13)
β_i	second element of column vector given by equation (3-13)
γ	angle of climb

LIST OF SYMBOLS (continued)

γ_i	third element of column vector given by equation (3-13)
δ_i	angle of flap rotation at strip i
δm	mass of a particle
θ ($\equiv q_{n+5}$)	orientation angle of no-deformation-body-fixed axes relative to constant-velocity axes
θ_i	orientation angle of strip-fixed axes relative to no-deformation-body-fixed axes
θ_{ui}	orientation angle of strip-fixed axes relative to unperturbed-strip-fixed axes
θ_f^{ij}	see equation (4-111)
ρ	air density
τ	see equation (4-87)
ϕ ($\equiv q_{n+4}$)	orientation angle of no-deformation-body-fixed axes relative to constant-velocity axes
ϕ_i	orientation angle of strip-fixed axes relative to no-deformation-body-fixed axes
ϕ_{ui}	orientation angle of strip-fixed axes relative to unperturbed-strip-fixed axes
$\chi_{if}, \chi_{iu}, \chi_{iw}$, $\chi_{iq}, \chi_{i\delta}$	coefficients in expression for local aerodynamic force (*cf* equation (4-15))
ψ ($\equiv q_{n+6}$)	orientation angle of no-deformation-body-fixed axes relative to constant-velocity axes
ψ_i	orientation angle of strip-fixed axes relative to no-deformation-body-fixed axes
ψ_{ui}	orientation angle of strip-fixed axes relative to unperturbed-strip-fixed axes
&	see equation (C-30)
$\&_{77}$	see equation (C-31)
∂	operator defined by equation (4-127)
ε	operator signifying the 'steady part of' (*cf* equation (4-126))
$\mathcal{L}_\phi$ etc	lower triangular matrix formed from the elements of $\{\phi\ \theta\ \psi\}$ etc - *cf* equation (4-95)

LIST OF SYMBOLS (continued)

Dressings

(i) Subscripts

A quantity is relative to or about the origin of a particular set of axes where

c denotes the constant-velocity axes

n denotes the no-deformation-body-fixed axes

s denotes the strip-fixed axes (this subscript is often omitted to avoid confusion with s for structural)

u denotes the unperturbed-strip-fixed axes

This subscript, when present, is always placed first.

i indicates point or force on ith strip

g indicates gravitational

p indicates propulsive

s indicates structural

0 indicates the strip $i = 0$ (the one containing the aircraft reference point), or a leading edge suctional force, or a quantity associated with the strip cg

The following two, when present, are always placed last except when followed by a nought.

e datum value

f value during datum motion

(ii) Superscripts

' applied to $\sum$ indicates a summation over the main part of a strip

" applied to $\sum$ indicates a summation over the flap part of a strip

* denotes certain modified coefficients

T denotes the transpose of a matrix

(2) denotes the two-dimensional value

(i) denotes the contribution from strip i

Other bracketed superscripts denote the axes of resolution, viz:

(c) constant-velocity axes

(dp) datum-path earth axes

(is) strip-fixed axes of the ith strip

LIST OF SYMBOLS (concluded)

(n)	no-deformation-body-fixed axes
(s)	strip-fixed axes (where it is clear which strip)
(us)	unperturbed-strip-fixed axes

(iii) Suprascripts

.	(dot)	denotes derivative with respect to time
-	(bar)	denotes typical or total
^	(circumflex)	refers to body-fixed axes, encastré modes, displacement body freedoms
~	(tilde)	see equation (A-8)

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No.	Author	Title, etc
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3	H.R. Hopkin	A scheme of notation and nomenclature for aircraft dynamics and associated aerodynamics. ARC R&M 3562 (1966)
4	D.L. Woodcock	Mathematical approaches to the dynamics of deformable aircraft - Part II The dynamics of deformable aircraft. ARC R&M 3776 (1971)
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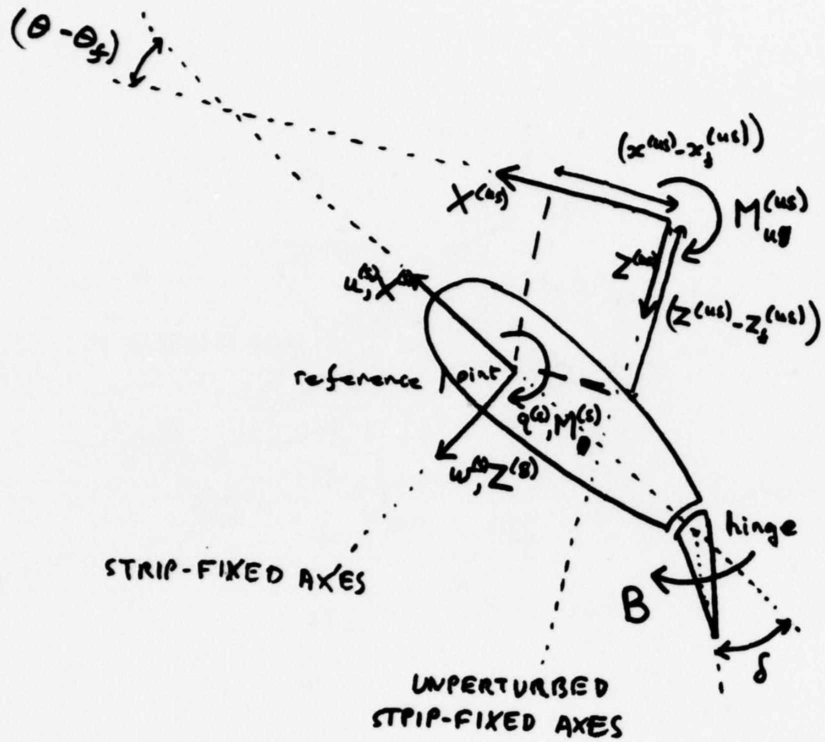
9	D.E. Davies	Generalised aerodynamic forces on a T-tail oscillating harmonically in subsonic flow. ARC R&M 3422 (1966)



u, w	linear velocities
q	angular velocities
X, Z	aerodynamic forces
M	aerodynamic moment
B	aerodynamic hinge moment
Subscript f	datum state value (NB $q_f = 0$)
Superscript (s)	strip-fixed axes
(us)	unperturbed strip-fixed axes

$$Z^{(s)} \approx Z_f^{(2)} + \hat{Z}_{\dot{x}}^{(2)}\left(u^{(s)} - u_f^{(us)}\right) + \hat{Z}_{\dot{z}}^{(2)}\left(w^{(s)} - w_f^{(us)}\right) + \hat{Z}_{\dot{\theta}}^{(2)} q^{(s)} + \hat{Z}_{\delta}^{(2)}(\delta - \delta_f)$$

$$Z^{(us)} \approx Z_f^{(2)} + Z_x^{(2)}\left(x^{(us)} - x_f^{(us)}\right) + Z_z^{(2)}\left(z^{(us)} - z_f^{(us)}\right) + Z_{\theta}^{(2)}(\theta - \theta_f) + Z_{\delta}^{(2)}(\delta - \delta_f)$$

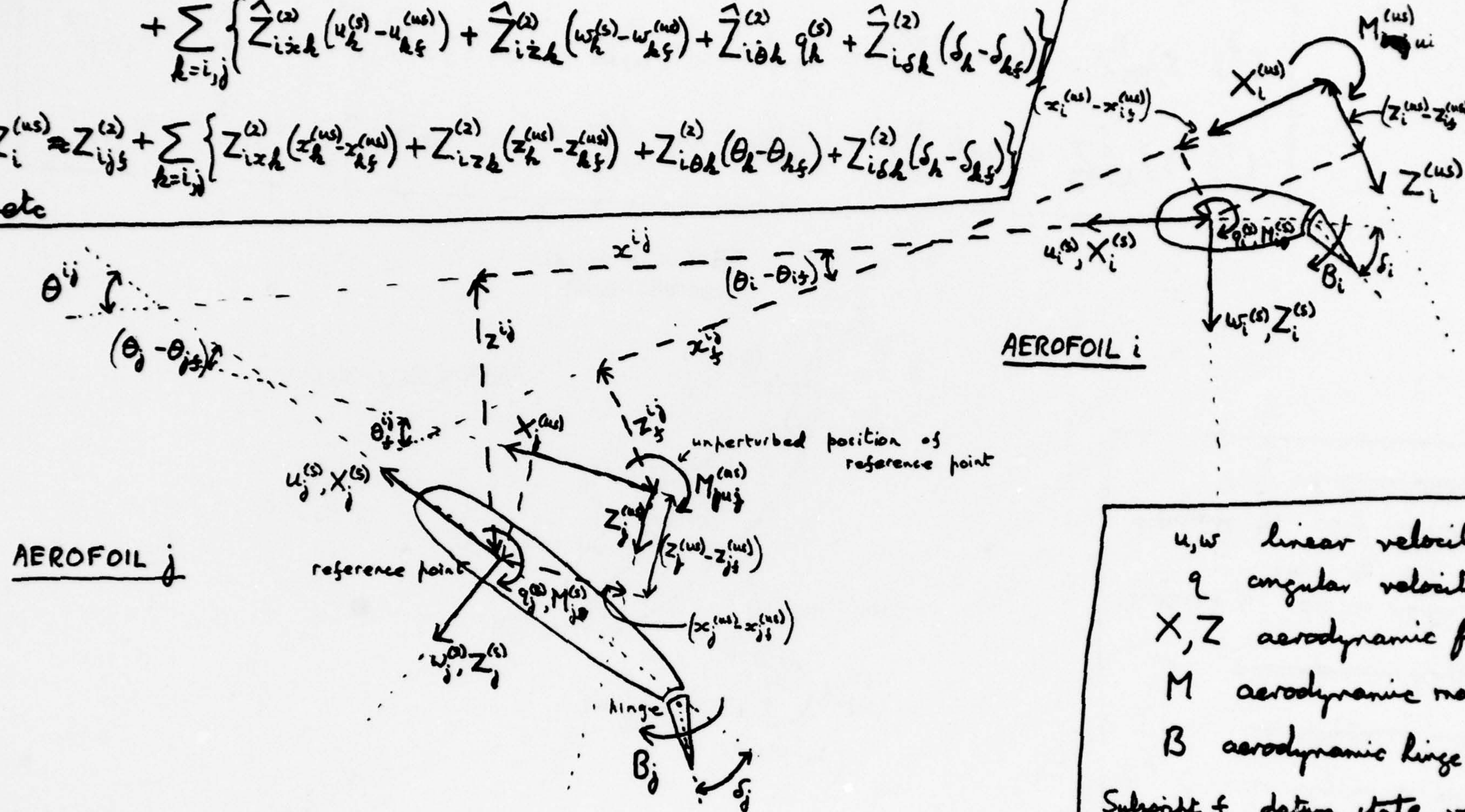
etc

Fig 1 Two-dimensional isolated aerofoil - notation

$$Z_i^{(s)} \approx Z_{ijf}^{(2)} + \hat{Z}_{ijx}^{(2)}\left(x^{ij}-x_f^{ij}\right) + \hat{Z}_{ijz}^{(2)}\left(z^{ij}-z_f^{ij}\right) + \hat{Z}_{ij\theta}^{(2)}\left(\theta^{ij}-\theta_f^{ij}\right)$$
$$+ \sum_{k=i,j}\left\{\hat{Z}_{ixk}^{(2)}\left(u_k^{(s)}-u_{kf}^{(us)}\right) + \hat{Z}_{izk}^{(2)}\left(w_k^{(s)}-w_{kf}^{(us)}\right) + \hat{Z}_{i\theta k}^{(2)} q_k^{(s)} + \hat{Z}_{i\delta k}^{(2)}\left(\delta_k-\delta_{kf}\right)\right\}$$

$$Z_i^{(us)} \approx Z_{ijf}^{(2)} + \sum_{k=i,j}\left\{Z_{ixk}^{(2)}\left(x_k^{(us)}-x_{kf}^{(us)}\right) + Z_{izk}^{(2)}\left(z_k^{(us)}-z_{kf}^{(us)}\right) + Z_{i\theta k}^{(2)}\left(\theta_k-\theta_{kf}\right) + Z_{i\delta k}^{(2)}\left(\delta_k-\delta_{kf}\right)\right\}$$

etc



u, w	linear velocities
q	angular velocities
X, Z	aerodynamic forces
M	aerodynamic moments
B	aerodynamic hinge moments

Subscript f datum state values
(NB $q_{if} = q_{jf} = 0$)

Fig 2 Two-dimensional tandem aerofoils - notation

REPORT DOCUMENTATION PAGE

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17. Abstract

A detailed formulation of the equations of motion of a deformable aircraft is given. The development is from Lagrange's equations for an inertial frame, and is made in terms of the position, orientation, force and inertia properties of narrow strips of the aircraft which lie fore and aft in the unperturbed state. The latter is one of constant linear velocity and zero angular velocity. Particular account is taken of the deformation and loading in the unperturbed state.

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